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Strong Cliques in Claw-Free Graphs

Michał Dębski^{1,2} · Małgorzata Śleszyńska-Nowak¹ 💿

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Abstract

For a graph G, $L(G)^2$ is the square of the line graph of G – that is, vertices of $L(G)^2$ are edges of G and two edges $e, f \in E(G)$ are adjacent in $L(G)^2$ if at least one vertex of e is adjacent to a vertex of f and $e \neq f$. The strong chromatic index of G, denoted by s'(G), is the chromatic number of $L(G)^2$. A strong clique in G is a clique in $L(G)^2$. Finding a bound for the maximum size of a strong clique in a graph with given maximum degree is a problem connected to a famous conjecture of Erdős and Nešetřil concerning strong chromatic index of graphs. In this note we prove that a size of a strong clique in a claw-free graph with maximum degree Δ is at most $\Delta^2 + \frac{1}{2}\Delta$. This result improves the only known result $1.125\Delta^2 + \Delta$, which is a bound for the strong chromatic index of claw-free graphs.

Keywords Strong chromatic index · Strong clique · Strong edge coloring

1 Introduction

For a graph G, by $L(G)^2$ we denote the square of the line graph of G – that is, vertices of $L(G)^2$ are edges of G and two edges $e, f \in E(G)$ are adjacent in $L(G)^2$ if at least one vertex of e is adjacent to a vertex of f and $e \neq f$. The strong chromatic index of G, denoted by s'(G), is the chromatic number of $L(G)^2$. This notion was introduced in 1983 by Fouquet and Jolivet [18].

Małgorzata Śleszyńska-Nowak m.sleszynska@mini.pw.edu.pl

Michał Dębski michal.debski87@gmail.com

¹ Faculty of Mathematics and Information Science, Warsaw University of Technology, Warsaw, Poland

² Masaryk University, Brno, Czech Republic

We are motivated by a line of research that asks for maximum possible value of the strong chromatic index of a graph with given maximum degree Δ . In 1985, Erdős and Nešetřil formulated the following conjecture [13]; see also [2].

Conjecture 1 If G is a graph with maximum degree Δ , then

$$s'(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{for even } \Delta, \\ \frac{5}{4}\Delta^2 - \frac{2\Delta - 1}{4}, & \text{for odd } \Delta. \end{cases}$$

Conjecture 1 would be sharp, as wittnessed by the graph obtained from 5-cycle by blowing each vertex up to an independent set of size roughly $\frac{4}{2}$ – note that the square of the line graph of such graphs is always a clique. Conversely, if $L(G)^2$ is a clique for some graph *G* of maximum degree Δ , then the number of edges of *G* is at most the bound from Conjecture 1, and equality holds only for aforementioned blowups of C_5 [8].

A simple greedy argument shows that $s'(G) \le 2\Delta^2$, and even a small improvement over the constant 2 is nontrivial. Molloy and Reed in 1997 gave an upper bound of $(2 - \epsilon)\Delta^2$ for $\epsilon \approx 0.002$ [26], which was subsequently improved by Bruhn and Joos [6] to $1.93\Delta^2$, by Bonamy, Perrett and Postle [5] to $1.835\Delta^2$, and recently by Hurley, de Joannis de Verclos and Kang [23] to $1.772\Delta^2$; those results apply only for sufficiently large Δ .

This problem has also been studied for restricted classes of graphs, including: bipartite graphs [15, 16], C_4 -free graphs [25], random graphs [17], graphs of bounded degeneracy [4, 7, 11], planar graphs [16, 22, 24], unit distance graphs [10] and graphs of maximum degree 3 and 4 [1, 9, 19–21]. In particular, the strong chromatic index of a claw-free graph with maximum degree Δ is at most $1.125\Delta^2 + \Delta$ [12].

We study a related problem of bounding the clique number of the square of the line graph. For a graph G, a *strong clique* in G is a clique in $L^2(G)$; the maximum size of a strong clique is $\omega(L^2(G))$. Proving an upper bound on the size of strong cliques should be easier than giving an upper bound on the strong chromatic index, because $\omega(L^2(G)) \leq s'(G)$, and it would provide supporting evidence for the conjecture of Erdős and Nešetřil; it is also a challenging problem on its own. Consider the following relaxation of Conjecture 1, posed by Faudree, Gyárfás, Schelp and Tuza [16]; see also [2].

Conjecture 2 If G is a graph with maximum degree Δ , then

$$\omega(L^2(G)) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{for even } \Delta, \\ \frac{5}{4}\Delta^2 - \frac{2\Delta - 1}{4}, & \text{for odd } \Delta. \end{cases}$$

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Conjecture 2 remains open, but is closer to being proved than Conjecture 1. The second author proved that $\omega(L^2(G)) \leq \frac{3}{2}\Delta^2$ [28], which was improved to $\frac{4}{3}\Delta^2$ by Faron and Postle [14]. There are also sharp results for specific classes of graphs: for bipartite graphs the upper bound is Δ^2 [16] (sharp for complete bipartite graphs), and for triangle-free graphs it is $\frac{5}{4}\Delta^2$ [3, Theorem 4.1.5] (sharp for blowups of C_5).

Since the strong chromatic index of claw-free graphs is at most $1.125\Delta^2 + \Delta$ [12], Conjecture 2 is obviously true for this class of graphs (for $\Delta \ge 12$ and $\Delta = 8, 10$). However, the constant 1.125 is probably far from optimal; we would like to know the sharp bound. Our contribution is that we reduce the upper bound to around Δ^2 .

Theorem 1 If G is a claw-free graph with maximum degree Δ , then

$$\omega(L^2(G)) \leq \Delta^2 + \frac{1}{2}\Delta.$$

2 Proof of Theorem 1

Let us define some basic notions. We say that two edges $e, f (e \neq f)$ of a graph G are *joined* in G if they are adjacent in G or there is some other edge which is adjacent to both of them in G. Note that in a strong edge-coloring of G any two edges which are joined must have distinct colors.

Let *G* be a graph and *S* be a subset of the vertex set of *G*. *G*[*S*] denotes the subgraph of *G* induced by *S*. We say that a graph *G* is *H*-free if *G* does not contain *H* as an induced subgraph. A *claw* is a complete bipartite graph $K_{1,3}$. $K_{1,3}$ -free graphs are called *claw*-free.

We start by proving auxiliary lemmas that are variations of Turan's Theorem, adapted to our setting. Proofs are inspired by the classic proofs of Turan's Theorem (the first one uses induction and the second one uses equivalence classes).

Lemma 1 Let A, B be disjoint sets such that $|A| \ge |B|$. Let G be a graph with the vertex set $V(G) = A \cup B$. Denote by E_A , E_B and E_{AB} respectively the set of edges with both endpoints in A, the set of edges with both endpoints in B and the set of edges with one endpoint in A and one endpoint in B; let e_A , e_B and e_{AB} be the size of the respective set.

If every two edges $e \in E_A$, $f \in E_B$ are joined, then $e_A + e_B - e_{AB} \leq {|A| \choose 2}$.

Proof We will proceed by induction on |B|. Note that the lemma is true for |B| < 2, because in this case $e_B = 0$ and $e_A \le {|A| \choose 2}$. Now suppose that the lemma is true if the two sets have size |A| - 2 and |B| - 2 respectively.

Let G be a graph that maximizes the sum $e_A + e_B - e_{AB}$. Consider two cases.

Case 1: $e_A = 0$ or $e_B = 0$. Note that $e_A + e_B \leq {\binom{|A|}{2}}$, because $|B| \leq |A|$, and $e_{AB} \geq 0$, so $e_A + e_B - e_{AB} \leq {\binom{|A|}{2}}$ as desired.

Case 2: $e_A > 0$ and $e_B > 0$. We claim that there are vertices $u, v \in A$ and $w, x \in B$ such that $uv, vx, wx \in E(G)$ and $uw, ux, vw \notin E(G)$. Indeed, if this was not the

case, then every edge inside A would be joined to every edge inside B by at least two edges, so removing from G an edge contributing to e_{AB} would not violate the required condition – contradicting the maximality of G.

Let $A' = A \setminus \{u, v\}$ and $B' = B \setminus \{w, x\}$. By $e_{A'}$, $e_{B'}$ and $e_{A'B'}$ we denote the number of edges with both endpoints in A', the number of edges with both endpoints in B' and the number of edges with one endpoint in A' and one endpoint in B', respectively. Also, for a vertex $y \in A' \cup B'$, by s(y) we denote the contribution of edges incident with y and a vertex from $\{u, v, w, x\}$ towards the sum $e_A + e_B - e_{AB} -$ that is, for $y \in A'$, s(y) is the number of neighbors of y in $\{u, v\}$ minus the number of neighbors of y in $\{w, x\}$ and for $y \in B'$, s(y) is the number of neighbors of y in $\{w, x\}$ minus the number of neighbors of y in $\{w, x\}$.

$$e_A + e_B - e_{AB} \le e_{A'} + e_{B'} - e_{A'B'} + \sum_{y \in A' \cup B'} s(y) + 1.$$
(1)

Now we claim that $s(y) \le 1$ for all $y \in A' \cup B'$. Suppose for the contrary that $s(y) \ge 2$ for some $y \in A'$. It implies that y is adjacent to both u and v and not adjacent to w and x – it follows that the edge uy is not joined to the edge wx, which contradicts the assumption on G. The same argument hold for $y \in B'$ when (u, v) and (w, x) are exchanged. Therefore, from inequality 1 we obtain

$$e_A + e_B - e_{AB} \le e_{A'} + e_{B'} - e_{A'B'} + |A| + |B| - 3.$$
⁽²⁾

By the induction assumption $e_{A'} + e_{B'} - e_{A'B'} \leq {\binom{|A|-2}{2}}$, therefore

$$e_A + e_B - e_{AB} \le \frac{(|A| - 2)(|A| - 3)}{2} + |A| + |B| - 3 = \binom{|A|}{2} + |B| - |A|, \quad (3)$$

which completes the proof by $|A| \ge |B|$.

Lemma 2 Let A, B be disjoint sets. If G is a graph with $V(G) = A \cup B$ such that

- (i) A does not contain an independent set of size 3,
- (ii) for every three vertices x, y, z, such that $x, y \in A$ and $z \in B$, if $xy \notin E(G)$, then $xz \in E(G)$ or $yz \in E(G)$,

then the number of edges incident to a vertex from A in G is at least $\begin{cases} \binom{a}{2} & \text{if } a < b, \\ \binom{a}{2} - \frac{(a-b)^2}{4} & \text{if } a \ge b. \end{cases}$

Proof Let A and B be fixed. Let G be a graph on $A \cup B$ that satisfies (i) and (ii) and has the minimum possible number of edges.

Claim There are no three vertices $u, v, w \in A$ such that $uv \notin E(G)$ and $uw, vw \in E(G)$.

Suppose for the contrary that there are three such vertices u, v, w. We will construct a graph G' with fewer edges, contradicting the minimality of G.

Case 1: Either *u* or *v* has smaller degree than *w*. Without loss of generality we assume that it is *u*. We obtain *G'* by removing all edges incident to *w* and adding an edge from *w* to each neighbor of *u* except *w* and to *u*. Note that $|E(G')| = |E(G)| - {}^{\circ}_{G}(w) + {}^{\circ}_{G}(u) < |E(G)|.$

Note that G' satisfies (i) – if there existed an independent set S of size 3 in G', then S would contain w, because only edges incident to w were changed, and would not contain u, because $uw \in E(G')$. But $S \setminus \{w\} \cup \{u\}$ would be an independent set in G of the same cardinality, which is a contradiction.

G' also satisfies (ii). To see this, consider a vertex $z \in B$ and $x \in A$ such that $xw \notin E(G')$. We have $xu \notin E(G)$, so either $xz \in E(G)$ or $uz \in E(G)$ – then we have $xz \in E(G')$ or $wz \in E(G')$ respectively.

Case 2: Both *u* and *v* have at least as large degree as *w*. In this case we obtain *G'* by removing all edges incident to *u* and *v*, adding edges from *u* and *v* to every neighbor of *w* and adding edges *uv*, *uw* and *vw*. Note that the degrees of *u*, *v* and *w* in *G'* are equal to ${}^{\circ}G(w)$, but an edge *uv* counts towards both ${}^{\circ}G'(u)$ and ${}^{\circ}G'(v)$. Therefore, we have $|E(G')| = |E(G)| - {}^{\circ}G(u) - {}^{\circ}G(v) + 2{}^{\circ}G(w) - 1 < |E(G)|$.

Note that G' satisfies (i) because – as in the previous case – for every independent set S in G' either $S \setminus \{u, v\} \cup \{w\}$ is an independent set in G of the same cardinality or S is disjoint from $\{u, v, w\}$.

G' satisfies (ii) because for every $z \in B$ and $x \in A$ such that $xu \notin E(G')$ or $xv \notin E(G')$ we have $xw \notin E(G)$, so either $xz \in E(G')$ or $uz, vz, wz \in E(G')$. This completes the proof of the claim.

Consider a relation $\sim \subseteq A^2$ such that $x \sim y$ iff $xy \in E(G)$ or x = y. From Claim 2 it follows that \sim is an equivalence relation on *A*. Since there are no independent sets of size 3 in *A*, \sim has at most two equivalence classes.

Let X be the second largest equivalence class of \sim – note that X may be an empty set. Set x := |X|, a := |A| and b := |B|.

Claim Each vertex from B has at least x neighbors in A.

Let $z \in B$. Note that either z is adjacent to all vertices from $A \setminus X$ – which gives at least $a - x \ge x$ neighbors of z in A – or z is not adjacent to some $v \in A \setminus X$. In the latter case z must be adjacent to all vertices from X by (ii), which again gives at least x neighbors in A – therefore, the proof of the claim is complete.

Now we can count the number of edges of *G* incident to a vertex from *A*. By the definition of *X*, there are exactly $\binom{x}{2} + \binom{a-x}{2}$ edges with both endpoints in *A* and, by Claim 2, there are at least *bx* edges from *A* to *B*, which totals to at least

$$f(x) := {a \choose 2} + x^2 + (b - a)x$$

edges. Note that *f* is a quadratic function in *x* that has minimum in point $x_{\min} = \frac{a-b}{2}$ and in increasing for $x > x_{\min}$. Therefore, we have

$$f(x) \ge \begin{cases} f(x_{\min}), & \text{for } x_{\min} \ge 0, \\ f(0), & \text{for } x_{\min} < 0. \end{cases}$$

Therefore, we get

$$f(x) \ge \begin{cases} \binom{a}{2} - \frac{(a-b)^2}{4}, & \text{for } a \ge b, \\ \binom{a}{2}, & \text{for } a < b, \end{cases}$$

which completes the proof.

Now we are ready to prove Theorem 1.

Proof of Theorem 1 Let $v_1, v_2 \in V(G)$ be vertices of G such that $e = v_1v_2 \in E(G)$ and e is an edge from a strong clique H with maximum size in G; we will think of H as a subgraph of G induced by the edges of the maximum strong clique.

Let D_1 be a set of vertices of G which are adjacent to v_1 in G and are not adjacent to v_2 in G, D_2 be a set of vertices of G which are adjacent to v_2 in G and are not adjacent to v_1 in G and D_3 be a set of vertices of G which are adjacent to both v_1 and v_2 in G (see Fig. 1). Let $|D_1| = d_1$, $|D_2| = d_2$, and $|D_3| = d_3$. We have:

$$d_1, d_2, d_3 \ge 0,$$
 (4)

$$d_1, d_2, d_3 \le \varDelta - 1, \tag{5}$$

$$d_1 + d_3 \le \varDelta - 1, \tag{6}$$

$$d_2 + d_3 \le \Delta - 1. \tag{7}$$

Without loss of generality, we can assume that $d_1 \ge d_2$.

Note that $G[D_1]$ is a clique – if there where two vertices $x, y \in D_1$, such that $xy \notin E(G)$, then edges xv_1, yv_1, v_1v_2 would form an induced claw in *G*. Similarly $G[D_2]$ is a clique. Moreover, $G[D_3]$ contains no independent set of size 3, as it would form an induced claw in *G* together with v1.

Note that every edge of *H* except *e* is joined to *e*, so it is incident to at least one vertex from $D_1 \cup D_2 \cup D_3$. Now we will count the number of edges in *H*. We claim that this number is at most



Fig. 1 Sets D_1, D_2, D_3

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$$\Delta(d_1 + d_2 + d_3) - {\binom{d_1}{2}} - 2{\binom{d_2}{2}} - {\binom{d_3}{2}} + 1 + I_{d_3 > d_1 + d_2} \frac{(d_3 - d_1 - d_2)^2}{4},$$
(8)

where $I_{d_3 > d_1+d_2}$ is 1 when $d_3 > d_1 + d_2$ and 0 otherwise. We obtain (8) as follows.

First we count the number of edges with exactly one endpoint in $D_1 \cup D_2$. All vertices from D_1 are incident to at most Δd_1 edges. We know that $G[D_1]$ is a clique, so all its edges have two endpoints in D_1 – we should not count them here, but we did it twice. Similarly for D_2 . In *H* there are also some edges with one endpoint in D_1 and one in D_2 . We counted them twice as well. Let $e_{D_1D_2}$ mean the number of such edges. Therefore the number of edges with exactly one endpoint in $D_1 \cup D_2$ is at most

$$\Delta(d_1 + d_2) - 2\binom{d_1}{2} - 2\binom{d_2}{2} - 2e_{D_1D_2}.$$
(9)

Now we add edges with both endpoints in $D_1 \cup D_2$. Let e_{D_1} mean the number of edges of H from $G[D_1]$, and e_{D_2} mean the number of edges of H from $G[D_2]$. The number of edges in H with both endpoints in $D_1 \cup D_2$ is

$$e_{D_1} + e_{D_2} + e_{D_1 D_2}. \tag{10}$$

The sum of (9) and (10) is

$$\Delta(d_1 + d_2) - 2\binom{d_1}{2} - 2\binom{d_2}{2} + e_{D_1} + e_{D_2} - e_{D_1D_2}.$$
 (11)

Let *S* be a graph with the vertex set $D_1 \cup D_2$ such that $xy \in E(S)$ if $x \in D_1, y \in D_2$ and $xy \in E(G)$ or $x, y \in D_i$ and $xy \in E(H)$. The graph *S* fulfills the assumptions of Lemma 1 with $A = D_1$ and $B = D_2$. So $e_{D_1} + e_{D_2} - e_{D_1D_2} \leq {d_1 \choose 2}$. Therefore (11) is at most

$$\Delta(d_1 + d_2) - \binom{d_1}{2} - 2\binom{d_2}{2}.$$
 (12)

Next we add to (12) edges from *H* that are incident to at least one vertex from D_3 ; there are at most Δd_3 of them. We obtain

$$\Delta(d_1 + d_2 + d_3) - \binom{d_1}{2} - 2\binom{d_2}{2}.$$
 (13)

Edges with one endpoint in $D_1 \cup D_2$ and one endpoint in D_3 and edges with both endpoints in D_3 are counted twice in (13) and will be subtracted. We can calculate their number by Lemma 2. Let *S* be a subgraph of *G* induced by $V(S) = D_1 \cup D_2 \cup D_3$. Note that for every three vertices $x, y \in D_3$ and $z \in D_1 \cup D_2$ if $xy \notin E(S)$, then $xz \in E(S)$ or $yz \in E(S)$, as it would form an induced claw together with *z*. Moreover, we have that D_3 contains no independent set of size 3. Thus the graph *S* fulfills the assumptions of Lemma 2 with $A = D_3$ and $B = D_1 \cup D_2$.

Therefore, the number of edges in question is at least $\binom{d_3}{2} - I_{d_3 > d_1 + d_2} \frac{(d_3 - d_1 - d_2)^2}{4}$. After subtracting it from (13) we obtain

$$\Delta(d_1 + d_2 + d_3) - {d_1 \choose 2} - 2{d_2 \choose 2} - {d_3 \choose 2} + I_{d_3 > d_1 + d_2} \frac{(d_3 - d_1 - d_2)^2}{4}.$$
 (14)

To obtain (8) we add 1 to (14) – because *e* has not been counted.

Claim The expression (8), subject to constraints (4, 5, 6, 7), is at most $\Delta^2 + \frac{1}{2}\Delta$.

We prove Claim 2 in the appendix.

By Claim 2 the proof is complete.

3 Further Research

We proved that the size of a strong clique in a claw-free graph with maximum degree Δ is at most $\Delta^2 + \frac{1}{2}\Delta$. The constant next to delta is less than $\frac{5}{4}$ from Conjecture 2 but we believe that the optimal constant is much smaller than that. A size of the biggest strong clique which we were able to construct is around $\frac{9}{16}\Delta^2$. The example for any Δ can be obtained from the graph G_1 , presented in Fig. 2, by blowing up each vertex to a clique of size $\frac{4}{4}$. In the constructed graph every two edges with endpoints in different cliques are joined, so all them create a strong clique. We want to construct a claw-free graph with a bigger strong clique or improve our upper bound.

We do not have any results for the maximum size of a strong clique in $K_{1,t}$ -free graphs with maximum degree Δ for t > 3. The only known result is $\left(2 - \frac{1}{t-2}\right)\Delta^2$, which is an upper bound for the strong chromatic index of $K_{1,t}$ -free graphs. Is it possible to tell something more about strong cliques in such graphs? Maybe for unit disks graphs? For them we know that the strong chromatic index is at most $1.625\Delta^2$.

Appendix: Proof of Claim 2

Claim 2 asserts that the following expression

Fig. 2 The graph G_1



$$\Delta(d_1 + d_2 + d_3) - {\binom{d_1}{2}} - 2{\binom{d_2}{2}} - {\binom{d_3}{2}} + 1 + I_{d_3 > d_1 + d_2} \frac{(d_3 - d_1 - d_2)^2}{4},$$
(8)

subject to constraints (4, 5, 6, 7), is at most $\Delta^2 + \frac{1}{2}\Delta$.

The partial derivative of (8) with respect to d_1 is equal to

$$\Delta - d_1 + \frac{1}{2} + I_{d_3 > d_1 + d_2} \frac{d_1 + d_2 - d_3}{2}.$$
(15)

If $d_3 \le d_1 + d_2$ then (15) is greater than 0 by (5). Otherwise (15) equals $\Delta + \frac{1}{2} + d_2 + d_3$, which is greater than 0 by (7). Thus (8) is increasing in d_1 . By (6) it follows that (8) is maximized for $d_1 = \Delta - d_3 - 1$, so it is at most

$$\begin{split} \mathcal{A}(\mathcal{A}-1+d_{2}) &- \left(\frac{\mathcal{A}-d_{3}-1}{2}\right) - 2\left(\frac{d_{2}}{2}\right) - \left(\frac{d_{3}}{2}\right) + 1 + I_{d_{3} > \mathcal{A}-d_{3}-1+d_{2}} \\ &\times \frac{(d_{3}-\mathcal{A}+d_{3}+1-d_{2})^{2}}{4} \\ &= \mathcal{A}(\mathcal{A}-1+d_{2}) - \frac{1}{2}(\mathcal{A}-d_{3}-1)^{2} + \frac{1}{2}(\mathcal{A}-d_{3}-1) - d_{2}^{2} \\ &+ d_{2} - \frac{1}{2}d_{3}^{2} + \frac{1}{2}d_{3} + I_{d_{3} > \frac{\mathcal{A}+d_{2}-1}{2}} \frac{(2d_{3}-d_{2}-\mathcal{A}+1)^{2}}{4} \\ &= -d_{2}^{2} + d_{2} + \mathcal{A}d_{2} - d_{3}^{2} - d_{3} + \mathcal{A}d_{3} + \frac{1}{2}\mathcal{A}^{2} + \frac{1}{2}\mathcal{A} \\ &+ I_{d_{3} > \frac{\mathcal{A}+d_{2}-1}{2}} \frac{(2d_{3}-d_{2}-\mathcal{A}+1)^{2}}{4}. \end{split}$$
(16)

Consider the following cases:

Case 1: $d_3 > \frac{4+d_2-1}{2}$, so

$$d_2 < 2d_3 - \varDelta + 1. \tag{17}$$

The partial derivative of (16) with respect to d_2 is

$$-2d_2 + 1 + \varDelta + \frac{1}{2}d_2 - d_3 + \frac{1}{2}\varDelta - \frac{1}{2} = -\frac{3}{2}d_2 - d_3 + \frac{3}{2}\varDelta + \frac{1}{2}, \quad (18)$$

which is greater than 0 by (7). Therefore, the maximum value of 18 is attained for maximum possible value of d_2 . We distinguish two cases.

Case 1.1: $2d_3 - \Delta + 1 \le \Delta - d_3 - 1$. Equivalently, we have

$$d_3 \le \frac{2\Delta - 2}{3}.\tag{19}$$

In this case we use upper bound on d_2 from the condition of Case 1, i.e. substitute $d_2 = 2d_3 - \Delta + 1$. It follows that (16) is at most

$$-(2d_3 - \Delta + 1)^2 + 2d_3 - \Delta + 1 + \Delta(2d_3 - \Delta + 1) - d_3^2 - d_3 + \Delta d_3 + \frac{1}{2}\Delta^2 + \frac{1}{2}\Delta = -5d_3^2 - 3d_3 + 7\Delta d_3 - \frac{3}{2}\Delta^2 + \frac{5}{2}\Delta.$$
(20)

The derivative of (20) with respect to d_3 is

$$-10d_3 - 3 + 7\Delta.$$
 (21)

It is greater than 0 by (19), so its maximum value is attained for $d_3 = \frac{2A-2}{3}$ and is equal to

$$-5\left(\frac{2\Delta-2}{3}\right)^{2} - 3\left(\frac{2\Delta-2}{3}\right) + 7\Delta\left(\frac{2\Delta-2}{3}\right) - \frac{3}{2}\Delta^{2} + \frac{5}{2}\Delta$$
$$= -\frac{20}{9}\Delta^{2} + \frac{40}{9}\Delta - \frac{20}{9} - \frac{25}{6}\Delta + 2 + \frac{19}{6}\Delta^{2}$$
$$= \frac{17}{18}\Delta^{2} + \frac{5}{18}\Delta - \frac{2}{9}.$$
 (22)

Case 1.2: $d_3 > \frac{2d-2}{3}$. In this case we use the inequality 7, i.e. substitute $d_2 = d - d_3 - 1$. It follows that (16) is at most

$$-(\varDelta - d_3 - 1)^2 + \varDelta - d_3 - 1 + \varDelta(\varDelta - d_3 - 1) - d_3^2 - d_3 + \varDelta d_3 + \frac{1}{2}\varDelta^2 + \frac{1}{2}\varDelta + \frac{(3d_3 - 2\varDelta + 2)^2}{4} = \frac{1}{4}d_3^2 - d_3 - \varDelta d_3 + \frac{3}{2}\varDelta^2 + \frac{1}{2}\varDelta - 1.$$
(23)

The derivative of (23) with respect to d_3 is

$$\frac{1}{2}d_3 - 1 - \varDelta. \tag{24}$$

It is smaller than 0 by (5), so its maximum value is attained for the smallest value of d_3 , $d_3 = \frac{2\Delta - 2}{3}$, and is equal to

$$\frac{1}{4} \left(\frac{2\Delta - 2}{3}\right)^2 - \frac{2\Delta - 2}{3} - \Delta \frac{2\Delta - 2}{3} + \frac{3}{2}\Delta^2 + \frac{1}{2}\Delta - 1$$

$$= \frac{1}{9}\Delta^2 - \frac{2}{9}\Delta + \frac{1}{9} - \frac{2}{3}\Delta + \frac{2}{3} - \frac{2}{3}\Delta^2 + \frac{2}{3}\Delta + \frac{3}{2}\Delta^2 + \frac{1}{2}\Delta - 1 \qquad (25)$$

$$= \frac{17}{18}\Delta^2 + \frac{5}{18}\Delta - \frac{2}{9}.$$

Case 2: $d_3 \leq \frac{\Delta + d_2 - 1}{2}$. Equivalently,

$$d_2 \ge 2d_3 - \varDelta + 1. \tag{26}$$

In this case $I_{d_3 > \frac{4+d_2-1}{2}}$ is 0, so (16) becomes

$$-d_{2}^{2} + d_{2} + \Delta d_{2} - d_{3}^{2} - d_{3} + \Delta d_{3} + \frac{1}{2}\Delta^{2} + \frac{1}{2}\Delta.$$
 (27)

Note that (27) is a sum of two functions

$$f_1(d_2) = -d_2^2 + d_2 + \Delta d_2 \tag{28}$$

and

$$f_2(d_3) = -d_3^2 - d_3 + \Delta d_3 + \frac{1}{2}\Delta^2 + \frac{1}{2}\Delta.$$
 (29)

The derivative of f_1 is equal to $-2d_2 + 1 + \Delta$. Thus the unique global maximum of f_1 is attained for $d_2 = \frac{\Delta+1}{2}$, and is equal to

$$-\frac{1}{4}\Delta^{2} - \frac{1}{2}\Delta - \frac{1}{4} + \frac{1}{2}\Delta^{2} + \frac{1}{2}\Delta + \frac{1}{2}\Delta + \frac{1}{2} = \frac{1}{4}\Delta^{2} + \frac{1}{2}\Delta + \frac{1}{4}.$$
 (30)

The derivative of f_2 is equal to $-2d_3 - 1 + \Delta$. Thus the unique global maximum of f_2 is attained for $d_3 = \frac{\Delta - 1}{2}$, and is equal to

$$-\frac{1}{4}\varDelta^{2} + \frac{1}{2}\varDelta - \frac{1}{4} + \frac{1}{2}\varDelta^{2} - \frac{1}{2}\varDelta - \frac{1}{2}\varDelta + \frac{1}{2} + \frac{1}{2}\varDelta^{2} + \frac{1}{2}\varDelta = \frac{3}{4}\varDelta^{2} + \frac{1}{4}.$$
 (31)

Therefore the unique maximum value of (27) is at most

$$\Delta^2 + \frac{1}{2}\Delta + \frac{1}{2}.$$
 (32)

Note that $d_2 = \frac{d+1}{2}$ and $d_3 = \frac{d-1}{2}$ do not satisfy (7). We know that (27) has a unique global maximum equal to (32), so the maximum value of (27) in d_2 , d_3 bounded by (7) is strictly smaller than (32). Moreover, we are interested only in Δ with integer values. Thus the maximum value of (27), constrained by (7), is at most

$$\Delta^2 + \frac{1}{2}\Delta. \tag{33}$$

Since the expressions (22), (25) and (33) are all smaller than or equal to $\Delta^2 + \frac{1}{2}\Delta$, the value of (8) is also at most $\Delta^2 + \frac{1}{2}\Delta$, which completes the proof of the claim.

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