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Research Article

Approximate Symmetries Analysis and Conservation Laws Corresponding to Perturbed Korteweg–de Vries Equation

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The Korteweg–de Vries (KdV) equation is a weakly nonlinear third-order differential equation which models and governs the evolution of fixed wave structures. This paper presents the analysis of the approximate symmetries along with conservation laws corresponding to the perturbed KdV equation for different classes of the perturbed function. Partial Lagrange method is used to obtain the approximate symmetries and their corresponding conservation laws of the KdV equation. The purpose of this study is to find particular perturbation (function) for which the number of approximate symmetries of perturbed KdV equation is greater than the number of symmetries of KdV equation so that explore something hidden in the system.

1. Introduction

Differential equations (DEs) are ubiquitous in modeling an extensive class of physical phenomena involving variation with respect to one or more independent variables. Therefore, DEs are broadly divided into ordinary DEs (ODEs) and partial DEs (PDEs). In different sectors of science and technology, PDEs have played a significant role. PDEs have numerous applications in mathematics, physics, fluid dynamics, mechanics, and physical chemistry. Modeling of PDEs under special conditions and constraints is advantageous in different situations for an effective manipulation of the varying phenomenon. The majority of real-world problems are almost nonlinear in nature, having no analytical solutions. In order to solve nonlinear problems, various approximations and techniques are used to gain high accuracy. In this regard, the approximate symmetry methods play a significant role. We have used the method of approximate Lie symmetry [1, 2], for PDEs to deal with the dynamical system more accurately. In the 1980s, the method of approximate Lie symmetry was developed by Baikov et al. [3, 4]. In obtaining the approximate solutions to such perturbed PDEs, the approximate symmetry method is an effective one. The extension of Lie's theory was mainly the basic reason behind the development of approximate symmetry, which deals with the systems by introducing small perturbation [5]. Symmetry applications to physical problems play a pivotal role in the development of conservation laws [6, 7]. The widely recognized KdV equation is a mathematical model for the depiction of weak nonlinear long wavelength waves in various branches of engineering and physics. It explains how waves evolve due to comparable effects of weak nonlinearity and dispersion. A perturbed nonlinear wave equation is a class of approximate symmetries which is computed using two newly developed methods. For both methods, the associated invariant solution with the approximate symmetries is constructed. By discussing the advantages and disadvantages of each method, the symmetries and solutions are compared. So, the Lie group technique in finding the exact solution of a differential equation has lost its importance. But an approximate Lie group technique has been implemented and used in

various methods for obtaining additional related information of differential equation. Perturbation analysis is one of the techniques which is used particularly for nonlinear systems.

This study is framed in the following manner: Section 2 is devoted to the development of exact symmetries and exact conservation laws of the KdV equation. The method to handle the approximate part of the KdV equation is developed in Section 3. The method so developed is applied to tackle the approximate part of the KdV equation for different cases and their corresponding conservation laws in Section 4. The work is concluded by describing the highlights in Section 5.

2. Exact Symmetries and Conservation Laws of the Korteweg–de Vries (KdV) Equation

The exact symmetries and conservation laws in the current study for the work considered in [8, 9] are worked out as follows:

The Korteweg–de Vries (KdV) equation which is a third-order nonlinear partial differential equation is

$$\mu_t - 6\mu\mu_x + \mu_{xxx} = 0. \tag{1}$$

The infinitesimal symmetry operator is

$$\begin{aligned}\mathbf{X}^{[3]} = {} & \phi\frac{\partial}{\partial x} + \varrho\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial \mu} + \varphi^{t}\frac{\partial}{\partial \mu_t} + \varphi^{x}\frac{\partial}{\partial \mu_x} + \varphi^{xx}\frac{\partial}{\partial \mu_{xx}} \\ & + \varphi^{xt}\frac{\partial}{\partial \mu_{xt}} + \varphi^{tt}\frac{\partial}{\partial \mu_{tt}} + \varphi^{xxx}\frac{\partial}{\partial \mu_{xxx}}.\end{aligned} \tag{2}$$

Applying this symmetry operator on (1),

$$\mathbf{X}^{[3]}\left(\mu_t - 6\mu\mu_x + \mu_{xxx}\right) = 0, \tag{3}$$

we get

$$\varphi^{t} - 6\mu\varphi^{x} - 6\varphi\mu_x + \varphi^{xxx} = 0. \tag{4}$$

The expanded form of equation (4) is

$$\begin{aligned}&\left[\varphi_t - \phi_t\mu_x + \left(\varphi_\mu - \varrho_t\right)\mu_t - \phi_\mu\mu_x\mu_t - \varrho\mu\mu_t^2\right] - 6\mu\left[\varphi_x + \left(\varphi_\mu - \phi_x\right)\mu_x - \varrho_x\mu_t - \phi_\mu\mu_x^2 - \varrho_x\mu_x\mu_t\right] \\ &- 6\varphi\mu_x + \left[\varphi xxx + \left(3\varphi_{\mu xx} - \phi_{xxx}\right)\mu_x - \varrho_{xxx}\mu_t + \left(3\varphi_{\mu\mu x} - 3\phi_{xx\mu}\right)\right. \\ &\mu_x^2 - 3\varrho_{\mu xx}\mu_x\mu_t + \left(3\varphi_{\mu x} - 3\phi_{xx}\right)\mu_{xx} - 3\varrho_{xx}\mu_{xt} + \left(\varphi_{\mu\mu\mu} - \phi_{\mu\mu x}\right)\mu_x^3 \\ &- 3\varrho_{\mu\mu x}\mu_x^2\mu_t + \left(3\varphi_{\mu\mu} - 9\phi_{\mu x}\right)\mu_x\mu_{xx} - 6\varrho_{x\mu}\mu_x\mu_{xt} - 3\varrho_{\mu x}\mu_t\mu_{xx} + \left(\varphi_\mu - 3\phi_x\right)\mu_{xxx} \\ &- 3\varrho_x\mu_{xxt} - \phi_{\mu\mu\mu}\mu_x^4 - \varrho_{\mu\mu\mu}\mu_x^3\mu_t - 6\phi_{\mu\mu}\mu_x^2\mu_{xx} - 3\varrho_{\mu\mu}\mu_x^2\mu_{xt} - 3\varrho_{\mu\mu}\mu_x\mu_t\mu_{xx} \\ &- 4\varphi_\mu\mu_x\mu_{xxx} - 3\varrho_\mu\mu_x\mu_{xxt} - 3\phi_\mu\mu_{xx}^2 - 3\varrho_\mu\mu_{xx}\mu_{xt} - \varrho_\mu\mu_t\mu_{xxx} = 0.\end{aligned} \tag{5}$$

Substituting equation (1) in equation (5), we get

$$\begin{aligned}&\varphi_t - \phi_t\mu_x + \left(\varphi_\mu - \varrho_t\right)\mu_t - \phi_\mu\mu_x\mu_t - \varrho_\mu\mu_t^2 - 6\varphi\mu_x - 6\mu\left[\varphi_x + \left(\varphi_\mu - \phi_x\right)\mu_x - \varrho_x\mu_t - \phi_\mu\mu_x^2 - \varrho_\mu\mu_x\mu_t\right] \\ &+ \varphi_{xxx} + \left(3\varphi_{xx\mu} - \phi_{xxx}\right)\mu_x - \varrho_{xxx}\mu_t + \left(3\varphi_{x\mu\mu} - 3\phi_{xx\mu}\right)\mu_x^2 \\ &- 3\varrho_{xx\mu}\mu_x\mu_t + \left(3\varphi_{x\mu} - 3\phi_{xx}\right)\mu_{xx} - 3\varrho_{xx}\mu_{xt} + \left(\varphi_{\mu\mu\mu} - \phi_{x\mu\mu}\right)\mu_x^3 - 3\varrho_{x\mu\mu}\mu_x^2\mu_t \\ &+ \left(3\varphi_{\mu\mu} - 9\phi_{x\mu}\right)\mu_x\mu_{xx} - 6\varrho_{x\mu}\mu_x\mu_{xt} - 3\varrho_{x\mu}\mu_t\mu_{xx} + 6\mu\left(\varphi_\mu - 3\phi_x\right)\mu_x - \left(\varphi_\mu - 3\phi_x\right)\mu_t \\ &- 3\varrho_x\mu_{xxt} - \phi_{\mu\mu\mu}\mu_x^4 - \varrho_{\mu\mu\mu}\mu_x^3\mu_t - 6\phi_{\mu\mu}\mu_x^2\mu_{xx} - 3\varrho_{\mu\mu}\mu_x^2\mu_{xt} - 3\varrho_{\mu\mu}\mu_x\mu_t\mu_{xx} \\ &- 24\mu\phi_\mu\mu_x^2 + 4\phi_\mu\mu_x\mu_t - 3\varrho_\mu\mu_x\mu_{xxt} - 3\phi_\mu\mu_{xx}^2 - 3\varrho_\mu\mu_{xx}\mu_{xt} - 6\mu\varrho_\mu\mu_x\mu_{xt} + \varrho_\mu\mu_t^2 = 0.\end{aligned} \tag{6}$$

Comparing the coefficients of various terms, we get the coefficients and monomials, as shown in Table 1.

Table 1 yields the required set of PDEs as follows:

$$\varrho_\mu = 0, \tag{7}$$

$$\phi_\mu = 0, \tag{8}$$

$$\varrho_x = 0, \tag{9}$$

$$3\phi_x - \varrho_t = 0. \tag{10}$$

Form (10),

$$\begin{aligned} \phi_{xx} &= 0 \\ \Rightarrow \varphi_{xx\mu} &= \phi_{xxx} \\ \Rightarrow \varphi_{xx\mu} &= 0, \\ \varphi &= \frac{-1}{6}\phi_t - 2\mu\phi_x. \end{aligned} \tag{11}$$

As

$$\phi_x = \frac{1}{3}\varrho_t, \tag{12}$$

therefore,

$$\begin{aligned} \varphi &= -\frac{1}{6}\phi_t - \frac{2}{3}\mu\varrho_t, \\ \varphi_t &= -\frac{1}{6}\phi_{tt} - \frac{2}{3}\mu\varrho_{tt}, \\ \varphi_x &= -\frac{1}{6}\phi_{xt} - \frac{2}{3}\mu\varrho_{tx}, \end{aligned} \tag{13}$$

$$\varphi_{xxx} = 0,$$

$$\phi_{tt} = 0, \tag{14}$$

$$\varrho_{tt} = 0. \tag{15}$$

Let

$$\begin{aligned} \varrho &= A(t) \\ \Rightarrow A_{tt}(t) &= 0. \end{aligned} \tag{16}$$

Integrating twice with respect to t yields

$$\begin{aligned} \Rightarrow A_t(t) &= k_1 \\ \Rightarrow A(t) &= k_1 t + k_2 \\ \Rightarrow \varrho &= k_1 t + k_2. \end{aligned} \tag{17}$$

From (10),

$$\begin{aligned} 3\phi_x - \varrho_t &= 0, \\ \phi_\mu = 0 &\Rightarrow \phi = B(x)t, \\ \phi_x &= \frac{1}{3}\varrho_t \\ \Rightarrow \phi_x &= \frac{1}{3}k_1. \end{aligned} \tag{18}$$

Integrating with respect to "x,"

$$\begin{aligned} \Rightarrow \phi &= \frac{1}{3}k_1 x + D(t) \\ \Rightarrow \phi_{tt} &= D_{tt}(t) = 0 \\ \Rightarrow D(t) &= k_3 t + k_4 \\ \Rightarrow \phi &= \frac{1}{3}k_1 x + k_3 t + k_4. \end{aligned} \tag{19}$$

From (13),

$$\begin{aligned} \varphi &= -\frac{1}{6}\phi_t - \frac{2}{3}\mu\varrho_t \\ &= -\frac{1}{6}k_3 - \frac{2}{3}k_1\mu. \end{aligned} \tag{20}$$

The general solution is

$$\begin{aligned} \varphi &= -\frac{2}{3}k_1\mu - \frac{1}{6}k_3, \\ \varrho &= k_1 t + k_2, \\ \phi &= \frac{1}{3}k_1 x + k_3 t + k_4. \end{aligned} \tag{21}$$

Hence, the Lie symmetry generators for the KdV equation are given, as shown in Table 2.

3. A New Procedure to Find the Approximate Symmetries

This section explains the development of the method for the approximate symmetries of the KdV equation. The KdV (1) is perturbed with the function $f(x,t,\mu(x,t),\mu(t,x))$ as

$$\mu_t - 6\mu\mu_x + \mu_{xxx} + \varepsilon f(x,t,\mu(x,t),\mu(t,x)) = 0, \tag{22}$$

where ε is a small parameter, causing the required perturbation in the KdV equation. The exact and approximate parts of (22) are

$$\begin{aligned} E_e &= \mu_t - 6\mu\mu_x + \mu_{xxx}, \\ E_a &= f(x,t,\mu(x,t)). \end{aligned} \tag{23}$$

Table 1: The exact symmetries of the given partial differential equation (PDE).

Coefficients	Monomials
$\varphi_t - 6\mu\varphi_x + \varphi_{xxx} = 0$	1
$-\phi_t - 6\varphi - 6\mu(\varphi_\mu - \phi_x) + 3\varphi_{x\mu} - \phi_{xxx} + 6\mu(\varphi_\mu - 3\phi_x) = 0$	μ_x
$-\phi_\mu + 6\mu\varrho_\mu - 3\varrho_{x\mu} + 4\phi_\mu = 0$	$\mu_x\mu_t$
$\varphi_\mu - \varrho_t + 6\mu\varrho_x - \varrho_{xxx} - (\varphi_\mu - 3\phi_x) = 0$	μ_t
$-\varrho_\mu + \varrho_\mu = 0$	μ_t^2
$6\mu\phi_\mu + 3\varphi_{x\mu\mu} - 3\phi_{xx\mu} - 24\mu\phi_\mu = 0$	μ_x^2
$3\varphi_{x\mu} - 3\phi_{xx} = 0$	μ_{xx}
$-3\varrho_{xx} = 0$	μ_{xt}
$\varphi_{\mu\mu\mu} - \phi_{x\mu\mu} = 0$	μ_x^3
$-3\varrho_{x\mu\mu} = 0$	$\mu_x^2\mu_t$
$3\varphi_{\mu\mu} - 9\phi_{x\mu} = 0$	$\mu_x\mu_{xx}$
$-6\varrho_{x\mu} - 6\mu\varrho_\mu = 0$	$\mu_x\mu_{xt}$
$-3\varrho_{x\mu} = 0$	$\mu_t\mu_{xx}$
$-3\varrho_x = 0$	μ_{xxt}
$\phi_{\mu\mu\mu} = 0$	μ_x^4
$\varrho_{\mu\mu\mu} = 0$	$\mu_x^3\mu_t$
$\phi_{\mu\mu} = 0$	$\mu_x^2\mu_{xx}$
$\varrho_{\mu\mu} = 0$	$\mu_x^2\mu_{xt}$
$\varrho_{\mu\mu} = 0$	$\mu_x\mu_t\mu_{xx}$
$\varrho_\mu = 0$	$\mu_x\mu_{xxt}$
$\phi_\mu = 0$	μ_{xx}^2
$\varrho_\mu = 0$	$\mu_{xx}\mu_{xt}$

Table 2: Lie symmetry generator of KdV equation.

Lie symmetry generators
$X_1 = (1/3)x(\partial/\partial x) + t(\partial/\partial t) - (2/3)\mu(\partial/\partial\mu)$
$X_2 = (\partial/\partial t)$
$X_3 = (\partial/\partial)$
$X_4 = (\partial/\partial x)$

Equation (22) can now be written in a more compact form as

$$E_e + \varepsilon E_a = 0. \tag{24}$$

On similar footing, we can combine the exact and approximate Lie symmetries as

$$\mathbf{X} = \mathbf{X}_e + \varepsilon\mathbf{X}_a. \tag{25}$$

Here,

$$\mathbf{X}_e = \phi_e\frac{\partial}{\partial x} + \varrho_e\frac{\partial}{\partial t} + \varphi_e\frac{\partial}{\partial\mu}, \tag{26}$$

is the exact Lie symmetry generator, and

$$\mathbf{X}_a = \phi_a\frac{\partial}{\partial x} + \varrho_a\frac{\partial}{\partial t} + \varphi_a\frac{\partial}{\partial\mu} \tag{27}$$

is the approximate Lie symmetry generator. Furthermore, ϕ, ϱ, and φ are the unknown functions of x, t, and μ, respectively.

Now, applying the generator $\mathbf{X}$ on (24), we have

$$(\mathbf{X}_e + \varepsilon\mathbf{X}_a)(E_e + \varepsilon E_a) = 0, \tag{28}$$

which yields

$$\mathbf{X}_e E_e + \varepsilon(\mathbf{X}_a E_e + \mathbf{X}_e E_a) + O(\varepsilon^2) = 0. \tag{29}$$

The comparison of coefficients of ε^0 and ε^1, respectively, yields the exact and approximate symmetries of the corresponding PDEs as in the following:

$$\begin{aligned} \mathbf{X}_e E_e &= 0, \\ \mathbf{X}_a E_e + \mathbf{X}_e E_a &= 0. \end{aligned} \tag{30}$$

The latter equation additionally gives the approximate Lie symmetries, which will not only provide the approximate conservation laws involved in the dynamics of the KdV equation but will also give the unknown function $f(x, t, \mu(x, t), \mu_i(t, x))$ [8, 10].

4. Approximate Symmetries and Corresponding Conservation Laws of the KdV Equation

In this section, we apply the developed method to find out the approximate symmetries. This method is applied and discussed for different cases. Considering the perturbed KdV equation [6, 11, 12],

$$\mu_t - 6\mu\mu_x + \mu_{xxx} + \varepsilon f(x, y, n, t, n_t, n_x, m_t, m_x) = 0. \tag{31}$$

By employing the method developed in [13–15] for the expansion of μ,

$$\mu = m + \varepsilon n. \tag{32}$$

Using this expansion in (31),

$$\begin{aligned}(m_t+\varepsilon n_t)-6(m+\varepsilon n)(m_x+\varepsilon n_x)+(m_{xxx}+\varepsilon n_{xxx})&=\varepsilon f\left(x,y,n,t,n_t,n_x,m_t,m_x\right),\\ m_t+\varepsilon n_t-6mm_x-6\varepsilon mn_x-6\varepsilon nm_x-6\varepsilon^2 nn_x+m_{xxx}+\varepsilon n_{xxx}&=\varepsilon f\left(x,y,n,t,n_t,n_x,m_t,m_x\right),\\ \left(m_t-6mm_x+m_{xxx}\right)+\varepsilon\left(n_t-6mn_x-6nm_x+n_{xxx}\right)+\varepsilon^2\left(-6nn_x\right)&=\varepsilon f\left(x,y,n,t,n_t,n_x,m_t,m_x\right).\end{aligned} \tag{33}$$

Equation (33) in more compact form is (neglecting higher power of ϵ)

$$\Delta_e+\varepsilon\Delta_a=0. \tag{34}$$

The comparison of the coefficients of ϵ^0 and ϵ^1 in (33) gives

$$\begin{aligned}\Delta_e &:= m_t-6mm_x+m_{xxx}=0,\\ \Delta_a &:= n_t-6mn_x-6nm_x+n_{xxx}-f\left(x,y,n,t,n_t,n_x,m_t,m_x\right)=0.\end{aligned} \tag{35}$$

The Lie symmetry generator is

$$\mathbf{X}=\mathbf{X}_e+\varepsilon\mathbf{X}_a=0. \tag{36}$$

Here,

$$\begin{aligned}\mathbf{X}_e&=\phi_e\frac{\partial}{\partial x}+\varrho_e\frac{\partial}{\partial t}+\varphi_e\frac{\partial}{\partial m}+\phi_e\frac{\partial}{\partial n},\\ \mathbf{X}_a&=\phi_a\frac{\partial}{\partial x}+\varrho_a\frac{\partial}{\partial t}+\varphi_a\frac{\partial}{\partial m}+\phi_a\frac{\partial}{\partial n}.\end{aligned} \tag{37}$$

Applying the Lie generator,

$$\begin{aligned}\mathbf{X}\left(\Delta_e+\varepsilon\Delta_a\right)&=0,\\ \left(\mathbf{X}_e+\varepsilon\mathbf{X}_a\right)\left(\Delta_e+\varepsilon\Delta_a\right)&=0,\end{aligned} \tag{38}$$

which gives us

$$\begin{aligned}\mathbf{X}_e\Delta_e+\varepsilon\left(\mathbf{X}_a\Delta_e+\mathbf{X}_e\Delta_a\right)+o\left(\varepsilon^2\right)&=0,\\ \mathbf{X}_e\Delta_e&=0,\\ \mathbf{X}_a\Delta_e+\mathbf{X}_e\Delta_a&=0.\end{aligned} \tag{39}$$

We now discuss the following cases in a bit detail.

Case I. Let

$$f\left(x,y,n,t,n_t,n_x,m_t,m_x\right)=-m_t-n_t. \tag{40}$$

Then, determining the system of PDEs from (35),

$$\begin{aligned}\varrho_{tt}&=0,\\ \varrho_m&=0,\\ \phi_t&=0,\\ \varrho_n&=0,\\ \phi_n&=0,\\ \phi_m&=0,\\ \varrho_x&=0,\\ \phi_x&=\frac{3}{\varrho_t},\\ \varphi&=\frac{-2}{3}m\varrho_t,\\ \phi&=-\frac{2}{3}\varrho_t n.\end{aligned} \tag{41}$$

As

$$\begin{aligned}\frac{\partial\varrho}{\partial m}&=0,\\ \frac{\partial\varrho}{\partial n}&=0,\\ \frac{\partial\varrho}{\partial x}&=0,\end{aligned} \tag{42}$$

which implies that "ϱ" is the function of "t" alone. Therefore,

$$\varrho_{tt} = 0. \tag{43}$$

Integrating the above equation twice with respect to "t" yields

$$\varrho = c_1 t + c_2. \tag{44}$$

Also,

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= 0, \\ \frac{\partial \phi}{\partial m} &= 0, \\ \frac{\partial \phi}{\partial n} &= 0, \end{aligned} \tag{45}$$

which shows that "ϕ" is the function of "x" alone. Therefore,

$$\frac{\partial \phi}{\partial x} = \frac{1}{3}\varrho t. \tag{46}$$

Putting the value of "ϱ_t" in (46), we get

$$\frac{\partial \phi}{\partial x} = \frac{1}{3}c_1. \tag{47}$$

Integrating (47), we get

$$\phi = \frac{1}{3}c_1 x + c_3. \tag{48}$$

Now,

$$\varphi = -\frac{2}{3}m\varrho_t. \tag{49}$$

Putting the value of "ϱ_t" in (49),

$$\varphi = -\frac{2}{3}mc_1. \tag{50}$$

By taking

$$\phi = -\frac{2}{3}\varrho_t n, \tag{51}$$

and putting the value of "ϱ_t" in (51),

$$\phi = -\frac{2}{3}c_1 n. \tag{52}$$

Therefore,

$$\begin{aligned} \phi &= \frac{1}{3}c_1 x + c_3, \\ \varrho &= c_1 t + c_2, \\ \varphi &= \frac{-2}{3}mc_1, \\ \phi &= \frac{-2}{3}c_1 n. \end{aligned} \tag{53}$$

The corresponding symmetry generators are tabulated in Table 3.

4.1. Conservation Laws. The conservation laws are developed as in the following:

$$\begin{aligned} \mathbf{X}_1\left(\psi\left(x, y, n, t, n_t, n_x, m_t, m_x\right)\right) &= 0, \\ \left(\frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2}{3}\frac{\partial}{\partial n} - \frac{2}{3}m\frac{\partial}{\partial m}\right)\psi &= 0, \\ \frac{1}{3}x\psi_x + t\psi_t - \frac{2}{3}\psi_n - \frac{2}{3}m\psi_m &= 0, \\ 3\frac{dx}{x} = \frac{dt}{t} = \frac{dn}{(-2/3)} = \frac{-3}{2}\frac{dm}{m} &= \frac{d\psi}{0}. \end{aligned} \tag{54}$$

Now, by taking

$$\begin{aligned} 3\frac{dx}{x} = \frac{dt}{t} &\Rightarrow x^3 = c_1 t \Rightarrow c_1 = \frac{x^3}{t}, \\ 3\frac{dx}{x} = \frac{dn}{(-2/3)} &\Rightarrow \ln x^3 = -\frac{2}{3}n + c_2 \Rightarrow c_2 = x^3 e^{(3/2)n}, \\ 3\frac{dx}{x} = \frac{-3}{2}\frac{dm}{m} &\Rightarrow x^3 = c_3 m^{-}\frac{3}{2} \Rightarrow c_3 = x^3 m^{(3/2)}, \\ \frac{dt}{t} = \frac{dn}{(-2/3)} &\Rightarrow \ln t = -\frac{2}{3}n + c_4 \Rightarrow c_4 = te^{(3/2)n}, \\ \frac{dt}{t} = \frac{-3}{2}\frac{dm}{m} &\Rightarrow t = c_5 m^{-}\frac{3}{2} \Rightarrow c_5 = tm\frac{3}{2}, \\ \frac{dn}{(-2/3)} = \frac{-3}{2}\frac{dm}{m} &\Rightarrow n + c_6 = \ln m \Rightarrow c_6 = me^{-n}, \end{aligned} \tag{55}$$

so

$$\begin{aligned} \psi = c_1 + c_2 + c_3 &= \frac{x^3}{t} + x^3 e^{(3/2)n} + x^3 m^{(3/2)} \\ &+ te^{(3/2)n} + tm\frac{3}{2} + me^{-n}. \end{aligned} \tag{56}$$

Furthermore,

$$\begin{aligned} X_2\left(\psi\left(x, y, n, t, n_t, n_x, m_t, m_x\right)\right) &= 0, \\ \psi_t = 0 &\Rightarrow \psi = c, \\ X_3\left(\psi\left(x, y, n, t, n_t, n_x, m_t, m_x\right)\right) &= 0, \\ \psi_x = 0 &\Rightarrow \psi = c. \end{aligned} \tag{57}$$

Following are the symmetries and their corresponding conservation laws of Case 1.

Case 2. Let

$$f\left(x, y, n, t, n_t, n_x, m_t, m_x\right) = -m_x. \tag{58}$$

Table 3: Lie symmetry generators.

Lie symmetry generators
$\mathbf{X}_1 = (1/3)x(\partial/\partial x) + t(\partial/\partial t) - (2/3)(\partial/\partial n) - (2/3)m(\partial/\partial m)$
$\mathbf{X}_2 = (\partial/\partial t)$
$\mathbf{X}_3 = (\partial/\partial x)$

From (35), we get after comparing the coefficients of ϵ^0 and ϵ^1,

$$\begin{aligned} m_t - 6mm_x + m_{xxx} &= 0, \\ n_t - 6mn_x - 6nm_x + n_{xxx} - m_x &= 0. \end{aligned} \tag{59}$$

Applying (36) to (59) yields the following system of PDEs:

$$\begin{aligned} \phi_t &= 0, \\ \phi_n &= 0, \\ \phi_m &= 0, \\ \phi_m &= 0, \\ \phi_n &= \frac{6\phi}{n+1}, \\ \phi_x &= \frac{1}{3}\varrho_t, \\ \phi_x &= 0, \\ \varrho_m &= 0, \\ \phi_{tt} &= 0, \\ \varrho_n &= 0, \\ \varphi &= \frac{-2}{3}\varrho_t m - \frac{1}{6}\phi_t \varrho_x = 0, \\ \varrho_{tt} &= 0. \end{aligned} \tag{60}$$

Solving the above system of PDEs, we get the following results:

$$\begin{aligned} \varphi &= \frac{-2}{3}c_1 m - \frac{1}{6}c_4, \\ \phi &= 6c_3 n + c_3, \\ \varrho &= c_1 t + c_2, \\ \phi &= \frac{1}{3}c_1 x + c_4 t + c_5. \end{aligned} \tag{61}$$

The approximate symmetries and their corresponding conservation laws in this case are given in Table 4.

Case 3. For this case, take

$$f(x, y, n, t, n_t, n_x, m_t, m_x) = -n_x. \tag{62}$$

From (35), we get after comparing the coefficients of ϵ^0 and ϵ^1,

$$\begin{aligned} m_t - 6mm_x + m_{xxx} &= 0, \\ n_t - 6mn_x - 6nm_x + n_{xxx} - n_x &= 0. \end{aligned} \tag{63}$$

This results in the following equations:

$$\begin{aligned} \phi_t &= 0, \\ \phi_{tt} &= 0, \\ \phi_m &= 0, \\ \phi_x &= 0, \\ \phi_n &= \frac{\phi}{n}, \\ \phi_n &= 0, \\ \phi_x &= 0, \\ \varrho_t &= 0, \\ \phi_m &= 0, \\ \varrho_m &= 0, \\ \varphi &= \frac{-1}{6}\phi_t \varrho_n = 0, \\ \varrho_x &= 0, \\ \varphi &= \frac{-1}{6}c_1, \\ \phi &= c_3 n, \\ \varrho &= c_4, \\ \phi &= c_1 t + c_2. \end{aligned} \tag{64}$$

Following are the symmetries and corresponding conservation laws of this Case 3.

Case 4. For this case, take

$$f(x, y, n, t, n_t, n_x, m_t, m_x) = mn, \tag{65}$$

then the system defined in (35) gives

$$\begin{aligned} m_t - 6mm_x + m_{xxx} &= 0, \\ n_t - 6mn_x - 6nm_x + n_{xxx} + mn &= 0. \end{aligned} \tag{66}$$

Applying (36) to (66), we get the following set of PDEs:

Table 4: Lie symmetry generators and corresponding conservation laws.

Lie symmetry generators	Corresponding conservation laws
$\mathbf{X}_1 = (1/3)x(\partial/\partial x) + t(\partial/\partial t) - (2/3)m(\partial/\partial m)$	$\psi_1 = (x^3/t) + (x\sqrt{m})^3 + tm^{(3/2)}$
$\mathbf{X}_2 = (\partial/\partial t)$	$\psi_2 = f(x, y, n, n_t, n_x, m_t, m_x)$
$\mathbf{X}_3 = (6n+1)(\partial/\partial n)$	$\psi_3 = g(x, y, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_4 = t(\partial/\partial x) - (1/6)(\partial/\partial m)$	$\psi_4 = (x/t) + 6m$
$\mathbf{X}_5 = (\partial/\partial x)$	$\psi_5 = h(y, n, t, n_t, n_x, m_t, m_x)$

$$\begin{aligned}
\varrho_t &= 0,\\
\phi_t &= \frac{1}{6}n\phi_t,\\
\phi_x &= 0,\\
\phi_m &= 0,\\
\varrho_x &= 0,\\
\phi_n &= \frac{\phi}{n},\\
\phi_{tt} &= 0,\\
\phi_x &= 0,\\
\varrho_m &= 0,\\
\phi_m &= 0,\\
\varrho_n &= 0,\\
\phi_n &= 0,\\
\varphi &= -\frac{1}{6}\phi_t.
\end{aligned} \tag{67}$$

The above equations yield

$$\begin{aligned}
\phi &= c_1 t + c_2,\\
\varrho &= c_3,\\
\varphi &= \frac{-1}{6}c_1,\\
\phi &= \frac{1}{6}n(c_1 t + 6c_4).
\end{aligned} \tag{68}$$

Following are the symmetries and corresponding conservation laws of Case 4.

Case 5. Let

$$f(x, y, n, t, n_t, n_x, m_t, m_x) = -n, \tag{69}$$

then the system defined in (35) gives

$$\begin{aligned}
m_t - 6mm_x + m_{xxx} &= 0,\\
n_t - 6mn_x - 6nm_x + n_{xxx} - n &= 0.
\end{aligned} \tag{70}$$

Applying (36) to (70) produces the following set of PDEs:

$$\begin{aligned}
\phi_t &= \varrho_t n,\\
\phi_n &= 0,\\
\phi_m &= 0,\\
\phi_m &= 0,\\
\phi_n &= \frac{\phi}{n},\\
\varrho_{tt} &= 0,\\
\phi_x &= 0,\\
\varrho_m &= 0,\\
\phi_x &= \frac{1}{3}\varrho_t\varrho_n = 0,\\
\varrho_x &= 0.
\end{aligned} \tag{71}$$

Solving the above equations, we get

$$\begin{aligned}
\phi &= \frac{c_1 x}{3} + tc_3 + c_4,\\
\varrho &= c_1 t + c_2,\\
\varphi &= \frac{-2}{3}m\varrho_t c_1 - \frac{1}{6}\phi_t c_3,\\
\phi &= n(c_1 t + c_5).
\end{aligned} \tag{72}$$

Following are the symmetries and corresponding conservation laws of Case 5.

Case 6. Assume

$$f(x, y, n, t, n_t, n_x, m_t, m_x) = -nm_t, \tag{73}$$

then the system defined in (35) gives

$$\begin{aligned}
m_t - 6mm_x + m_{xxx} &= 0,\\
n_t - 6mn_x - 6nm_x + n_{xxx} - nm_t &= 0.
\end{aligned} \tag{74}$$

Applying (36) to (74) results in the following set of PDEs:

Table 5: Lie symmetry generators and corresponding conservation laws.

Lie symmetry generators	Corresponding conservation laws
$\mathbf{X}_1 = (1/3)x(\partial/\partial x) + t(\partial/\partial t) + nt(\partial/\partial n) - (2/3)m(\partial/\partial m)$	$\psi_1 = (x^3/t) + x^3 + x^3 m^{(3/2)} + e^t n + tm^{(3/2)}$
$\mathbf{X}_2 = (\partial/\partial t)$	$\psi_2 = f(x, y, n, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_3 = (\partial/\partial x) - (1/6)(\partial/\partial m)$	$\psi_3 = x + 6m$
$\mathbf{X}_4 = (\partial/\partial x)$	$\psi_4 = g(y, n, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_5 = n(\partial/\partial n)$	$\psi_5 = h(x, y, t, n_t, n_x, m_t, m_x)$

Table 6: Lie symmetry generator and corresponding conservation laws.

Lie symmetry generators	Corresponding conservation laws
$\mathbf{X}_1 = (1/3)x(\partial/\partial x) + t(\partial/\partial t) - (2/3)(\partial/\partial n) - (2/3)m(\partial/\partial m)$	$\psi_1 = (x^3/t) + x^3 e^{(3/2)} n + x^3 m^{(3/2)} + te^{(3/2)} n + tm^{(3/2)} + me^{-n}$
$\mathbf{X}_2 = (\partial/\partial t)$	$\psi_2 = f(x, y, n, n_t, n_x, m_t, m_x)$
$\mathbf{X}_3 = (\partial/\partial x)$	$\psi_3 = g(y, n, t, n_t, n_x, m_t, m_x)$

Table 7: Lie symmetry generators and corresponding conservation laws.

Lie symmetry generators	Corresponding conservation laws
$\mathbf{X}_1 = (-1/6)(\partial/\partial x) + t(\partial/\partial t)$	$\psi_1 = (e^{-6x}/t)$
$\mathbf{X}_2 = x(\partial/\partial x)$	$\psi_2 = f(y, n, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_3 = n(\partial/\partial n)$	$\psi_3 = g(x, y, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_4 = (\partial/\partial t)$	$\psi_4 = h(x, y, n, n_t, n_x, m_t, m_x)$

Table 8: Lie symmetry generator and corresponding conservation laws.

Lie symmetry generators	Corresponding conservation laws
$\mathbf{X}_1 = t(\partial/\partial x) + (1/6)nt(\partial/\partial n) - (1/6)(\partial/\partial m)$	$\psi_1 = (e^{(x/t)}/n^6) + ne^m + (x/t) + 6m$
$\mathbf{X}_2 = (\partial/\partial x)$	$\psi_2 = f(y, n, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_3 = (\partial/\partial t)$	$\psi_3 = g(x, y, n, n_t, n_x, m_t, m_x)$
$\mathbf{X}_4 = n(\partial/\partial n)$	$\psi_4 = h(x, y, t, n_t, n_x, m_t, m_x)$

Table 9: Lie symmetry generators and corresponding conservation laws.

Lie symmetry generators	Corresponding conservation laws
$\mathbf{X}_1 = n(\partial/\partial n)$	$\psi_1 = f(x, y, t, n_t, n_x, m_t, m_x)$
$\mathbf{X}_2 = (\partial/\partial t)$	$\psi_2 = g(x, y, n, n_t, n_x, m_t, m_x)$
$\mathbf{X}_3 = (\partial/\partial x)$	$\psi_3 = h(y, n, t, n_t, n_x, m_t, m_x)$

$$\begin{aligned}
\varrho_{tx} &= 0,\\
\phi_t &= 0,\\
\phi_x &= 0,\\
\phi_m &= 0,\\
\phi_n &= 0,\\
\phi_n &= \frac{\phi}{n},\\
\phi_m &= 0,\\
\varrho_n &= 0,\\
\phi_x &= 0,\\
\varrho_t &= 0,\\
\phi_t &= 0,\\
\varrho_m &= 0.
\end{aligned} \tag{75}$$

Solving the above set of equations, we get

$$\begin{aligned}
\phi &= c_3,\\
\varrho &= c_2,\\
\varphi &= 0,\\
\phi &= c_1 n.
\end{aligned} \tag{76}$$

Following are the symmetries and corresponding conservation laws of Case 6.

5. Conclusion

The KdV equation is a 3rd order nonlinear partial differential equation which is modeled for waves on the surface of shallow water. It admits four Lie symmetries given in Table 2. In this paper, approximate symmetry techniques are used for finding some classes of the KdV equations that admit more symmetries as compared to the exact KdV equations. We perturbed the KdV equation by different particular functions and found the corresponding Lie symmetries. We found two important classes for the perturbed KdV equation that admits five Lie symmetries. The Lie symmetries along with their conservation laws are given in Tables 2, 4 and 5. In both the tables, we have an extra symmetry which corresponds to an extra conservation law. This extra conservation law is an extra information hidden in the system, the perturbation procedure explored it. Sometimes, the symmetry does not exist for the exact equation, but perturbation enables the equation to admit a symmetry. We saw this phenomenon in this research work by comparing Tables 2–6. Table 1 contains the determining PDEs which provide the set of Lie symmetries admitted by the given PDE. We have 4 Lie symmetries given in Table 2 for exact PDE, while Tables 3 and 6 contain only three Lie symmetries; in these cases, we lose one symmetry (one conservation law). Tables 7 and 8 consist of four Lie symmetries which means that all the conservation laws are recovered in these cases. Table 9 includes the lie symmetry generators and corresponding conservation laws.

Data Availability

There are no specific data used in the study of this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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