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Symmetric solutions of the singular minimal surface equation

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Abstract
We classify all rotational symmetric solutions of the singular minimal surface equation in both cases $\alpha < 0$ and $\alpha > 0$. In addition, we discuss further geometric and analytic properties of the solutions, in particular stability, minimizing properties and Bernstein-type results.

Keywords Singular minimal surface equation · Symmetric solutions · Stability

Mathematics Subject Classification 49Q05 · 53A10

1 Introduction

The singular (or symmetric) minimal surface equation (in short: s.m.s.e.) is the equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{\alpha}{u\sqrt{1+|Du|^2}} \tag{1}$$

where $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open, denotes some function and $\alpha \in \mathbb{R}$ stands for some real number. Observe that for any $\alpha \in \mathbb{R}$ equation (1) is the Euler equation of the variational integral

$$E_\alpha(u) := \int_\Omega u^\alpha \sqrt{1+|Du|^2}\, \mathrm{d}x, \tag{2}$$

which is the nonparametric counterpart of the energy functional

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$$\mathcal{E}_\alpha(M) := \int\limits_M x_{n+1}^\alpha \, \mathrm{d}\mathcal{H}_n, \tag{3}$$

where $M \subset \mathbb{R}^n \times \mathbb{R}^+$ denotes some C^2-hypersurface and $\mathcal{H}_n$ stands for the n-dimensional Hausdorff measure. M is stationary for the energy (3), i.e., the first variation $\delta\mathcal{E}_\alpha(M, \xi) = 0$ for all vector fields ξ with compact support, if the mean curvature $H(x)$ of M at $x = (x_1 ... x_{n+1})$ satisfies

$$H(x) = \alpha \frac{\nu_{n+1}}{x_{n+1}} \tag{4}$$

where $\nu = (\nu_1, ..., \nu_{n+1})$ denotes the unit normal of M at x.

There are several key motivations to considering equation (1) and the associated variational integral (2) in both cases $\alpha > 0$ and $\alpha < 0$, respectively. On the one hand, the variational problem connected to the integral (2) is a singular problem which admits in general only $\frac{1}{2}$-Hölder continuous solutions, cp. Bemelmans, Dierkes [1], Dierkes [3], Tennstädt [13], and on the other hand (1) with $\alpha = 1$ appears as a model problem for the multi-dimensional analogue of the catenary, while, for $\alpha = m \in \mathbb{N}$, equation (1) describes symmetric minimal hypersurfaces in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$. Finally, for $\alpha = -n$ equation (1) is the minimal surface equation in the hyperbolic space $\mathbb{R}^n \times \mathbb{R}^+$ of curvature $K = -1$. A further application in architecture lends special interest to the problem. We refer the reader to Dierkes [7, 8] for more detailed information and further references to the literature.

Clearly, the case $\alpha = 0$ corresponds to classical minimal surfaces and will not be considered here.

In this paper we are mainly interested in the study of "rotational symmetric solutions of the s.m.s.e." (1), that is in solutions $u = u(r)$, $r := |x|$ of the equation

$$\frac{u''(r)}{\left(1 + u'(r)^2\right)^{\frac{3}{2}}} + (n-1)\frac{u'(r)}{r\sqrt{1 + u'(r)^2}} = \frac{\alpha}{u(r)\sqrt{1 + u'(r)^2}}. \tag{5}$$

which can be transformed into

$$\left(\frac{r^{n-1}u'(r)}{\sqrt{1 + u'(r)^2}}\right)' = r^{n-1}\frac{\alpha}{u(r)\sqrt{1 + u'(r)^2}} \tag{6}$$

together with the initial condition $u(0) = z_0 > 0$ and $u'(0) = 0$. Additionally we are also interested in parametric rotational symmetric solutions of equation (4). Let $\gamma(s) = (x(s), y(s))$, $s \in I \subset \mathbb{R}$, be a C^2 -parametrization by arc length of a curve γ, which upon rotation about the $y = x_{n+1}$ - axis defines a C^2 surface $M \subset \mathbb{R}^{n+1}$. Then M is a solution of (4) if γ satisfies

$$y''(s)x'(s) - y'(s)x''(s) + (n-1)\frac{y'(s)}{x(s)} = \alpha\frac{x'(s)}{y(s)}. \tag{7}$$

Introducing the tangent angle $\psi \in C^1(I)$ by

$$\tan\psi = \frac{y'}{x'} \quad or \quad \left(x', y'\right) = (\cos\psi, \sin\psi), \tag{8}$$

equation (7) can be transformed into the system

$$\begin{cases} x'(s) = \cos\psi(s) \\ y'(s) = \sin\psi(s) \\ \psi'(s) + (n-1)\dfrac{\sin\psi(s)}{x(s)} = \alpha\dfrac{\cos\psi(s)}{y(s)} \end{cases} \tag{9}$$

Further insights are obtained if we also consider the stationarity condition in the "phase space" (ψ, θ) , where $\theta \in C^1(I)$ stands for the polar angle

$$\tan\theta = \frac{y}{x}. \tag{10}$$

Then we have

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}\psi} =& \frac{\mathrm{d}\theta}{\mathrm{d}s}\cdot\frac{\mathrm{d}s}{\mathrm{d}\psi} = \sin\theta\cos\theta\frac{\tan\psi-\tan\theta}{\alpha-(n-1)\tan\psi\tan\theta} \\ =& \sin\theta\cos\theta\frac{\sin(\psi-\theta)}{\alpha\cos\psi\cos\theta-(n-1)\sin\psi\sin\theta}, \end{aligned} \tag{11}$$

which in turn leads to the planar ordinary system

$$\begin{cases} \dfrac{\mathrm{d}\psi}{\mathrm{d}t} = \alpha\cos\psi\cos\theta-(n-1)\sin\psi\sin\theta \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \sin\theta\cos\theta\sin(\psi-\theta) \end{cases} \tag{12}$$

Symmetric solutions of (1) and (4) resp., i.e., solutions of (6), (7), (9), (11) and (12), respectively, have been considered in the papers by Keiper [10] (preprint), Lopez [11] and Dierkes [3, 4, 7]. The reason why we start a new discussion is twofold: Firstly, we add some new solutions to (6) or (9) by carefully analyzing system (12) in the phase space (cp. Theorem 1, (iii) and Theorem 7). Secondly, we accomplish Keiper's [10] paper by thoroughly examining, extending and proving his assertions and indications of proofs (cp. Theorem 14). In addition to that we completely classify the stability and (non-) minimizing properties of the solutions (cp. Theorems 17 and 20) and prove a Bernstein-type result (cp. Theorem 19) (Fig. 1). Finally, we think that our arguments are slightly cleaner than the original proofs in [10] and [11].

2 Solutions for $\alpha < 0$

For negative α, symmetric solutions of the s.m.s.e (1) are classified as follows, cp. also Lopez [11].

Theorem 1 *Let* $\gamma = (x(s), y(s)) \subset \mathbb{R}\times\mathbb{R}^+, s \in I$, *denote a maximal solution of* (7), $\alpha < 0$, *then* γ *can be described by one of the following three cases where* (*iii*) *only occurs if* $-1 < \alpha < 0$:

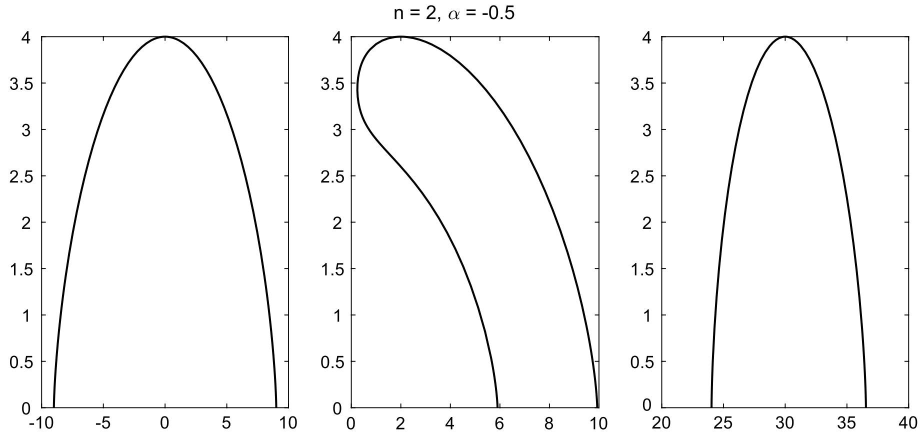

Fig. 1 All three possible types of solutions for $\alpha < 0$ as described in Theorem 1, from left to right

(i) *γ is the graph of a strictly concave symmetric function on a bounded interval of the x-axis which attains its maximum at $x = 0$ and intersects the x-axis orthogonally.*

(ii) *γ stays on one side of the y-axis (on $x > 0$, say) and intersects the x-axis orthogonally in both end points. $x = x(s)$ attains exactly one interior minimum in the interval I and no maximum. γ has a horizontal tangent at the unique maximum of $y = y(s)$. Furthermore, γ has no self-intersections.*

(iii) *γ stays on one side of the y-axis (on $x > 0$, say) and is the graph of a strictly concave function, which is defined over some compact interval of the x-axis. At both end points γ intersects the x-axis orthogonally.*

We split the proof of Theorem 1 in five Lemmata and start with some simple properties of solutions of system (9):

Lemma 2 *If $\gamma = (x, y)$ is a maximal solution that intersects the y-axis, then γ can be written as the graph of a strictly concave function on a bounded interval of the x-axis that is symmetric about the y-axis. The function has a unique maximum at $x = 0$. At the boundary points a, b of I, γ intersects the x-axis.*

Proof Without loss of generality, let $x(0) = 0$, $x'(0) = 1$, and $\psi(0) = 0$. (x, y, ψ) solves the following initial value problem on $I = (a, b)$:

$$\begin{cases} x'(s) = \cos\psi(s) \\ y'(s) = \sin\psi(s) \\ \psi'(s) + (n-1)\dfrac{\sin\psi(s)}{x(s)} = \alpha\dfrac{\cos\psi(s)}{y(s)} \\ x(0) = 0, y(0) = y_0 > 0, \psi(0) = 0 \end{cases} \tag{13}$$

The mirror image $\left(\widetilde{x}, \widetilde{y}, \widetilde{\psi}\right) := (-x(-s), y(-s), -\psi(-s))$ yields another solution to (13) and, by uniqueness, γ must be symmetric about the y-axis. From now on, we only have to discuss $\gamma(s)$ for $s \geq 0$.

De L'Hôpital's rule yields $\lim_{s\to 0}\frac{\sin\psi(s)}{x(s)} = \lim_{s\to 0}\frac{\cos\psi(s)\psi'(s)}{\cos\psi(s)} = \psi'(0)$ and therefore $\psi'(0) = \frac{\alpha}{ny_0} < 0$, after taking the limit in (13). $y'(0) = 0$ and $y''(0) = \psi'(0) < 0$ further imply that y attains a local maximum at $s = 0$.

Assume there exists a smallest $s_1 \in (0, b)$ with $\psi'(s_1) = 0$. Then $\psi''(s_1) \geq 0$ and, by taking the derivative in (13),

$$\psi''(s) + (n-1)\cos\psi(s)\frac{\psi'(s)x(s)-\sin\psi(s)}{x(s)^2} = \alpha\sin\psi(s)\frac{-\psi'(s)y(s)-\cos\psi(s)}{y(s)^2}.$$

Plugging in s_1, we obtain $0 \leq \psi''(s_1) = \sin\psi(s_1)\cos\psi(s_1)\left(\frac{n-1}{x(s_1)^2} - \frac{\alpha}{y(s_1)^2}\right)$, and it follows that $\psi(s_1) \in \left[0, \frac{\pi}{2}\right] + \pi\mathbb{Z}$.

Independently of the existence of s_1, now assume there is a smallest $s_2 \in (0, b)$ with $\psi(s_2) = -\frac{\pi}{2}$. Then (13) implies $\psi'(s_2) = \frac{n-1}{x(s_2)} > 0$.

The existence of either s_i implies the existence of the other s_j in $(0, s_i)$, leading to a contradiction. Hence, we must have $\psi'(s) < 0$ and $\psi(s) \in \left(-\frac{\pi}{2}, 0\right)$ for all $s \in (0, b)$ and γ is the graph of a strictly concave function. It remains to determine the behavior of $\gamma(s)$ near $s = b$.

Since x and y are monotone and bounded and s is the arc length, it follows that $b < \infty$. Because b is maximal, (x, y, ψ) has to leave every compact subset of $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ near b. The only way this is possible is that $y(b) := \lim_{s\to b} y(s) = 0$. Furthermore, the limits $x(b) := \lim_{s\to b} x(s) > 0$ and $\psi(b) := \lim_{s\to b}\psi(s) \in \left[-\frac{\pi}{2}, 0\right)$ exist, concluding the proof. $\square$

Before proving $\psi(b) = -\frac{\pi}{2}$, we discuss different types of solutions.

Lemma 3 *If a maximal solution γ has a horizontal tangent at a point $(x_0, y_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ (w. l. o. g. at $s = 0$ and $\psi(0) = 0$), then y attains a global maximum there and the right branch of γ $(s > 0)$ can be written as the graph of a strictly concave function on a bounded interval of the x-axis that intersects the x-axis at $s = b$. For the left branch $(s < 0)$, there are two possibilities*:

(a) *There is a point $s = s_2 < 0$ where x takes on a global minimum. For $s_2 < s < 0$, γ is the graph of a strictly concave function on an interval of the x-axis. For $a < s < s_2$, with decreasing s, γ runs through an inflection point (maximum of ψ) and then toward the x-axis.*

(b) *Also the left branch of γ is the graph of a strictly concave function on a bounded interval of the x-axis.*

Proof From (9), we get $\psi'(0) = \frac{\alpha}{y_0} < 0$. Because of $y'(0) = \sin\psi(0) = 0$ and $y''(0) = \cos\psi(0)\psi'(0) < 0$, y attains a local maximum at 0. Hence, the behavior of the right branch of γ can be discussed as in Lemma 2.

For negative s close to zero, we have ψ strictly decreasing, $\psi(s) \in \left(0, \frac{\pi}{2}\right)$ and $x(s) > 0$.

Assume the existence of a maximal $s_1 \in (a, 0)$ with $\psi(s_1) \in \left(0, \frac{\pi}{2}\right)$ and $\psi'(s_1) = 0$. Then $\psi''(s_1) \leq 0$, but also

$\psi''(s_1) = \sin\psi(s_1)\cos\psi(s_1)\left(\frac{n-1}{x(s_1)^2} - \frac{\alpha}{y(s_1)^2}\right) > 0$, which is a contradiction.

If $\psi(s) < \frac{\pi}{2}$ for all $s \in (a, 0)$, the behavior of γ is described by case (b).

Otherwise, there must exist a maximal $s_2 \in (a, 0)$ with $\psi(s_2) = \frac{\pi}{2}$. There we have $\psi'(s_2) = \frac{1-n}{x(s_2)} < 0$, so ψ is still decreasing. Because of $x'(s_2) = 0$ and $x''(s_2) = -\psi'(s_2) > 0$, x has a local minimum at s_2.

Assume there is a maximal $s_3 \in (a, s_2)$ such that $\psi(s_3) = \pi$. Then $\psi'(s_3) \le 0$, but by (9) also $\psi'(s_3) = \frac{-\alpha}{y(s_3)} > 0$, leading to a contradiction.

Assume now that there is a maximal $s_4 \in (a, s_2)$ with $\psi(s_4) = \frac{\pi}{2}$. Then $\psi'(s_4) \ge 0$, but on the other hand $\psi'(s_4) = \frac{1-n}{x(s_4)} < 0$, another contradiction.

Therefore $\psi(s) \in \left(\frac{\pi}{2}, \pi\right)$ for all $s \in (a, s_2)$. Next, we will show that ψ' does not stay negative on all of I. Assuming otherwise that $\psi'(s) < 0$ for all $s \in (a, s_2)$, consider the following cases:

Case $a > -\infty$: Because $I = (a, b)$ is maximal, we can deduce just like in Lemma 2 that the limits $y(a) := \lim_{s\to a} y(s) = 0$, $x(a) > 0$ and $\psi(a) \in \left(\frac{\pi}{2}, \pi\right]$ exist. However, $\lim_{s\to a} \psi'(s) = \lim_{s\to a} \alpha\frac{\cos\psi(s)}{y(s)} - \lim_{s\to a}(n-1)\frac{\sin\psi(s)}{x(s)} = \infty$, contradicting $\psi'(s) < 0$ for all $s \in (a, s_2)$.

Case $a = -\infty$: The limits $y(a) \ge 0$ and $\psi(a) \in \left(\frac{\pi}{2}, \pi\right]$ exist by monotonicity and boundedness. Because γ is parametrized by arc length, we must also have $\psi(a) = \pi$ and $\lim_{s\to a} x(s) = \infty$. Using (9), we obtain $\lim_{s\to a}\psi'(s) = \lim_{s\to a}\alpha\frac{\cos\psi(s)}{y(s)} \in (0, \infty]$ which is again a contradiction to $\psi'(s) < 0$ for all $s \in (a, s_2)$.

This yields the existence of a point $s_5 \in (a, s_2)$ with $\psi'(s_5) = 0$. Since the second derivative $\psi''(s_5) = \sin\psi(s_5)\cos\psi(s_5)\left(\frac{n-1}{x(s_5)^2} - \frac{\alpha}{y(s_5)^2}\right) < 0$, we also know that $\psi'(s) > 0$ for all $s \in (a, s_5)$.

As in Lemma 2, it follows that $a > -\infty$, $y(a) := \lim_{s\to a} y(s) = 0$, $x(a) > 0$ and $\psi(s) \in \left[\frac{\pi}{2}, \pi\right)$. This finishes the discussion of case (a).

□

From now on, we will focus on the trajectories of the system

$$\begin{cases} \dfrac{\mathrm{d}\psi}{\mathrm{d}t} = \alpha\cos\psi\cos\theta - (n-1)\sin\psi\sin\theta \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \sin\theta\cos\theta\sin(\psi - \theta) \end{cases} \tag{12}$$

and the singular points $(\psi, \theta) = \left(0, \frac{\pi}{2}\right)$ and $\left(-\frac{\pi}{2}, 0\right)$. All other singular points can be obtained through translations by multiples of π. By the Hartman–Grobman theorem (see, e.g., Perko [12, Section 2.8]), it is sufficient to analyze the linearized system in order to understand the qualitative behavior of trajectories near the equilibrium points. At the point $(\psi, \theta) = \left(0, \frac{\pi}{2}\right)$, we find

$$\begin{pmatrix} 1-n & -\alpha \\ 0 & 1 \end{pmatrix}$$

as the matrix of the linearized system. The eigenvalue $1 - n < 0$ has eigenvector $(1, 0)$ while the value 1 corresponds to the eigenvector $\left(1, -\frac{n}{\alpha}\right)$, whence the point $\left(0, \frac{\pi}{2}\right)$ is a saddle point of (12) with stable manifold in $(1,0)$-direction and unstable manifold in $\left(1,-\frac{n}{\alpha}\right)$ -direction.

For $(\psi,\theta)=\left(-\frac{\pi}{2},0\right)$, we have the linearized system with matrix

$$\begin{pmatrix} \alpha & n-1 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

Hence, $(-\frac{\pi}{2},0)$ is a stable node. The eigenvectors corresponding to -1 and α are $(1,0)$ and $\left(1,-\frac{\alpha+1}{n-1}\right)$, respectively. Therefore, the behavior depends on the size of α:

$\underline{\alpha<-1:}$ All trajectories with $\theta\neq 0$ have direction $\left(1,-\frac{\alpha+1}{n-1}\right)$.

$\underline{-1<\alpha<0:}$ All trajectories have direction $(1,0)$, except for a single one with direction $\left(-1,\frac{\alpha+1}{n-1}\right)$.

$\underline{\alpha=-1:}$ The eigenspace is one-dimensional. The trajectories spiral toward the ψ -axis in negative direction.

By symmetry, $(\psi,\theta)=\left(\frac{\pi}{2},0\right)$ is an unstable node.

Lemma 4 *Every maximal solution of (9) in the quadrant $\theta\in\left(0,\frac{\pi}{2}\right)$ intersects either the x- and y-axis or the x-axis in the respective endpoints at a right angle. In particular this also holds in case (b) of Lemma 3.*

Proof We look at the region $(\psi,\theta)\in\mathbb{R}\times\left(0,\frac{\pi}{2}\right)$ in the phase plane of system 12 and show that all trajectories start and end in a singular (or equilibrium) point. Clearly, it suffices to consider the asymptotic behavior as $t\to\infty$.

Assume on the contrary, there is a trajectory not approaching a singular point as $t\to\infty$. Then, by the Poincaré–Bendixson theorem (see Hartman [9, Section VII.4]) it must approach a periodic orbit or is periodic itself. This orbit must contain a critical point on the inside (see Corollary 2 in Perko [12, Section 3.12]). However, there is no critical point in $(\psi,\theta)\in\mathbb{R}\times\left(0,\frac{\pi}{2}\right)$; hence, we obtain a contradiction.

In the particular case of the left branch of a solution of type (b) in Lemma 3, the only possible end point is of type $(\psi,\theta)=\left(\frac{\pi}{2},0\right)$ because of the monotonicity of ψ. Also x cannot converge to zero as $y\to 0$, since for small $s\to-\infty$ the trajectory has small values of $\theta>0$ and ψ is close to $\frac{\pi}{2}$.

Finally, every trajectory that is not of type as in Lemma 2 starts in an unstable and ends in a stable node (Fig. 2), so there always exists a (finite) point with $\psi\in\pi\mathbb{Z}$. □

It still remains to show that both cases in Lemma 3 are possible:

Lemma 5 *There exists a value $\theta_0=\theta_0(n,\alpha)\in\left[0,\frac{\pi}{2}\right)$ such that for every maximal solution (x,y,ψ) of (9) in the region $\theta\in\left(0,\frac{\pi}{2}\right)$ with $\psi(0)=0$, we have:*

If $\theta(0)>\theta_0$, then $\gamma=(x,y)$ is of type (a) in Lemma 3.

If $\theta(0)\leq\theta_0$, then $\gamma=(x,y)$ is of type (b) in Lemma 3.

Furthermore, $\theta_0=0$ if and only if $\alpha\leq -1$.

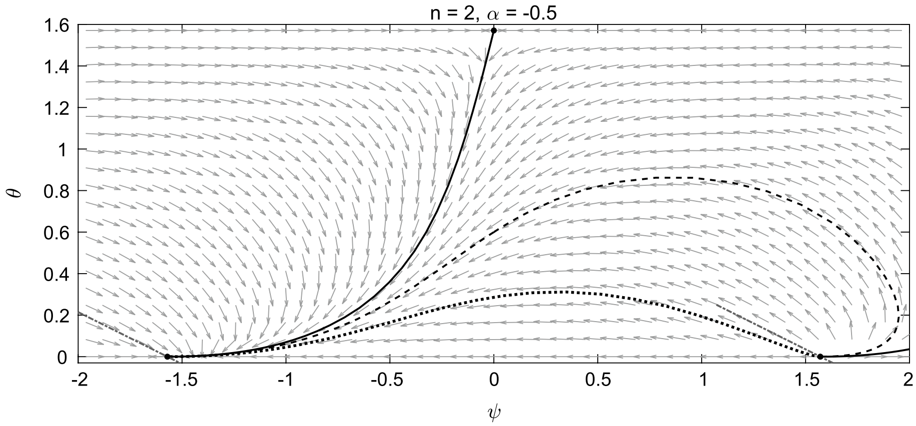

Fig. 2 ψ-θ-phase plane of (12) for $-1 < \alpha < 0$. Solutions of types (i)–(iii) are depicted by the solid, dashed and dotted curves, respectively. The dashed-dotted line indicates which slope coming out of $(\psi, \theta) = \left(\frac{\pi}{2}, 0\right)$ dictates whether the solution is of type (ii) or (iii)

Proof A trajectory (x, y, ψ) is of type (a) if and only if there is a $s_2 \in (a, 0)$ with $\psi\left(s_2\right) = \frac{\pi}{2}$. Recall the behavior near $(\psi, \theta) = \left(\frac{\pi}{2}, 0\right)$ in the following cases:

Case $\alpha < -1$:

The trajectory $(\psi(t), \theta(t))$ of γ leaves $\left(\frac{\pi}{2}, 0\right)$ in the direction $\left(1, -\frac{\alpha+1}{n-1}\right)$. Therefore, we have $\psi(t) > \frac{\pi}{2}$ for small $t > -\infty$ and γ is described by (a). Whence we have $\theta_0 = 0$.

Case $\alpha = -1$:

Same conclusion as in the previous case, this time with direction $(1, 0)$. Again γ is of type (a) and $\theta_0 = 0$.

Case $-1 < \alpha < 0$:

There is exactly one trajectory ϕ that leaves $\left(\frac{\pi}{2}, 0\right)$ in direction $\left(-1, \frac{\alpha+1}{n-1}\right)$. Since we then have $\psi(t) < \frac{\pi}{2}$ for small $t > -\infty$ and the value $\psi = \frac{\pi}{2}$ cannot be assumed later, the solutions belonging to ϕ must be of type (b). Furthermore, the value $\theta_0 \in \left(0, \frac{\pi}{2}\right)$ is uniquely determined by the requirement that ϕ passes through the point $(\psi, \theta) = \left(0, \theta_0\right)$. Every trajectory with $\theta(t) < \theta_0$ at the point t where $\psi = 0$ has to leave $\left(\frac{\pi}{2}, 0\right)$ in direction $(-1, 0)$ and therefore corresponds to (b). Otherwise, if $\theta > \theta_0$ at $\psi = 0$, it leaves the starting point with direction $(1, 0)$, and the solutions belong to case (a). □

Finally, also solutions of type (a) have no self-intersections.

Lemma 6 *For every solution $\gamma = (x, y)$, we have $\gamma\left(s_1\right) \neq \gamma\left(s_2\right)$ for all $a \leq s_1 < s_2 \leq b$, where we have set $\gamma(a) := \lim_{s \to a} \gamma(s)$ and $\gamma(b) := \lim_{s \to b} \gamma(s)$.*

Proof Assume this is not the case and choose a minimal s_2 such that there is a (maximal) $s_1 < s_2$ with $a \leq s_1 < s_2 \leq b$ and $\gamma\left(s_1\right) = \gamma\left(s_2\right)$. Then $\theta\left(s_1\right) = \theta\left(s_2\right)$ and $\left|\psi\left(s_1\right) - \psi\left(s_2\right)\right| \geq \pi$.

Recall that the trajectories of solutions that touch the y-axis divide the phase plane in regions of ψ-width equal to π, and these trajectories meet every value of $\theta \in \left(0, \frac{\pi}{2}\right)$ exactly once. In other words, the existence of a solution γ and points $s_1, s_2 \in I = (a, b)$ as above is impossible.

We still have to consider the case $s_1 = a$ and $s_2 = b$. Since γ orthogonally meets the x-axis at both points, we must have $\left|\psi\left(\tilde{s}_1\right) - \psi\left(\tilde{s}_2\right)\right| > \pi$ for $\tilde{s}_i$ sufficiently close to s_i, and we can pick the $\tilde{s}_i$ such that $\theta\left(\tilde{s}_1\right) = \theta\left(\tilde{s}_2\right)$. Again, this leads to a contradiction. □

The proof of Theorem 1 now easily follows from Lemma 2–6.

The solutions of type (i) can be used in conjunction with a maximum principle to show that (1) has no solutions defined on all of $\mathbb{R}^n$ for $\alpha < 0$. Also type (i) solutions minimize locally the energy $E_\alpha(\cdot)$ in suitable classes of BV-functions (see Dierkes [7]).

3 Solutions for $\alpha > 0$

The first part of this section is based on the unpublished work by Keiper [10] who investigated the case $\alpha = 1$. In addition to carefully scrutinizing his results, we give more detailed arguments for the proofs and generalize to arbitrary $\alpha > 0$. Also we adopt Keiper's notation to denote smooth entire solutions of the system (15) as the n-α-tectum (the Latin word for "roof").

3.1 Classification and geometry

As before, we consider the system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} = \cos\psi, & \dfrac{\mathrm{d}y}{\mathrm{d}s} = \sin\psi \\ \dfrac{\mathrm{d}\psi}{\mathrm{d}s} = \alpha\dfrac{\cos\psi}{y} - (n-1)\dfrac{\sin\psi}{x} & \end{cases} \tag{15}$$

as well as system (12). In analogy to Keiper, we refer to any solution which intersects the y-axis orthogonally as the n-α-tectum (which is unique up to homotheties) (Fig. 3, left).

All solutions of (15) are classified in the following Theorem (Figs. 3, 4):

Theorem 7 *Let $\alpha > 0$, then every maximal solution $\gamma = \gamma(s)$ of (15) is of one of the following types*:

(i) *n-α-tectum: γ is the graph of a symmetric function on the x-axis that assumes a global minimum at $x = 0$ and is strictly increasing for $x > 0$. As $x \to \infty$, the function is asymptotic to the straight line $\sqrt{\frac{\alpha}{n-1}}x$.*

(ii) *n-α-cone: γ lies on one side of the y-axis (say, $x > 0$) and is the ray through the origin with gradient $\sqrt{\frac{\alpha}{n-1}}$.*

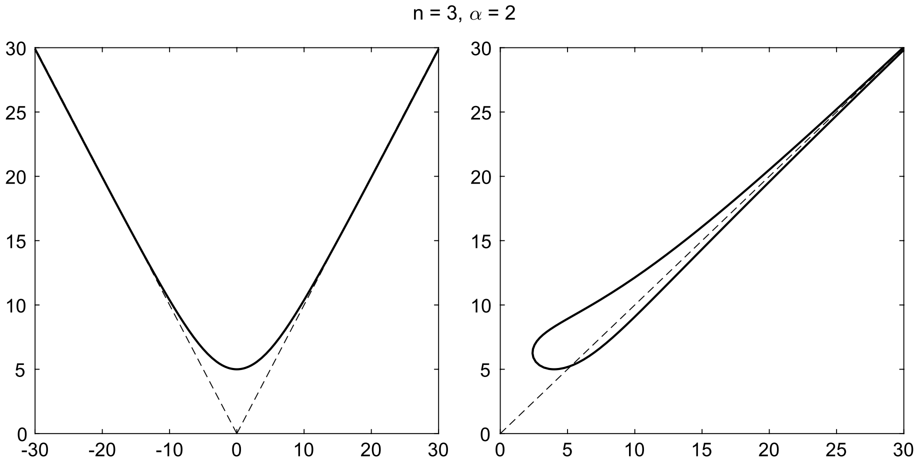

Fig. 3 Solutions to (15) of type (i) (left) and of type (iv) (right) as described in Theorem 7. The dotted lines represent the n-α-cone (type (ii))

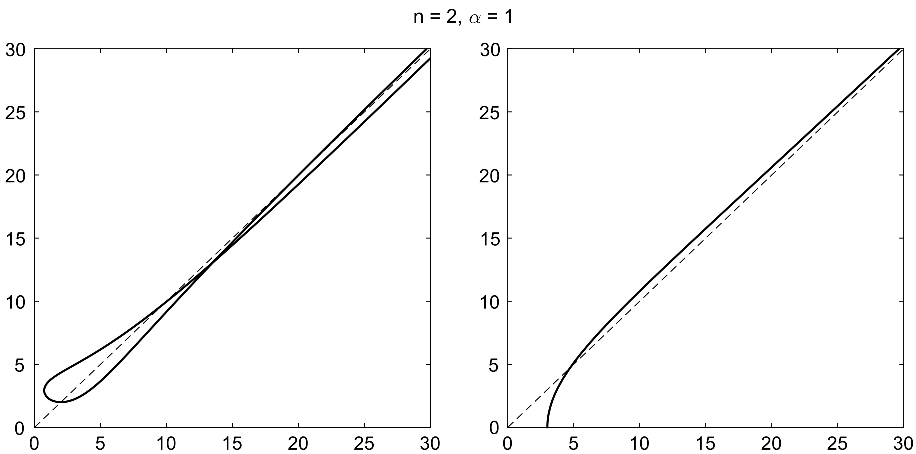

Fig. 4 Solutions to (15) of type (iii) (right) and of type (iv) (left) as described in Theorem 7. The dotted lines represent the n-α-cone (type (ii))

(iii) *γ lies on one side of the y-axis (say, $x > 0$) and is the graph of a strictly increasing function on an interval $\left(x_0, \infty\right)$ of the x-axis with $x_0 > 0$. At the point $\left(x_0, 0\right)$, γ orthogonally meets the x-axis. As $x \to \infty$, the function is asymptotic to the straight line $\sqrt{\frac{\alpha}{n-1}}x$.*

(iv) *γ lies on one side of the y-axis (say, $x > 0$) and both ends are asymptotic to the line $\sqrt{\frac{\alpha}{n-1}}x$. Furthermore, we have $\left|\lim_{s\to\infty} \psi(s) - \lim_{s\to-\infty} \psi(s)\right| = \pi$.*

We start our analysis with the n-α-tectum.

Lemma 8 *The n-α-tectum is the graph of a symmetric function on some interval of the x-axis which assumes a global minimum at $x = 0$ and is strictly increasing for $x > 0$. $\gamma(s)$ is defined for every $s \in \mathbb{R}$ and is unbounded.*

Proof Without loss of generality, let $x(0) = 0$ and $\psi(0) = 0$. By symmetry, we only have to consider $s \geq 0$.

At $s = 0$, we get $\frac{d\psi}{ds} = \frac{\alpha}{ny} > 0$ from (15) and therefore y has a local minimum there and $\psi \in \left(0, \frac{\pi}{2}\right)$ for small $s > 0$.

With the same line of arguments as in the previous section, it is easy to see that ψ stays in the interval $\left(0, \frac{\pi}{2}\right)$ for all $s > 0$. In particular, $\frac{dx}{ds} > 0$ and $\frac{dy}{ds} > 0$ for all $s > 0$.

Since (x, y, ψ) is a maximal solution to (15) and γ is parametrized by arc length, γ must be defined for all $s \in \mathbb{R}$ and γ is unbounded. □

For $\alpha > 0$, the system

$$\begin{cases} \dfrac{d\psi}{dt} = \alpha \cos\psi \cos\theta - (n-1)\sin\psi \sin\theta \\ \dfrac{d\theta}{dt} = \sin\theta \cos\theta \sin(\psi - \theta) \end{cases} \tag{12}$$

has the following four types of singular points,

$$\begin{array}{llrl} \text{(A)} & \psi = & k_1\pi, & \theta = \frac{\pi}{2} + k_2\pi \\ \text{(B)} & \psi = & \frac{\pi}{2} + k_1\pi, & \theta = k_2\pi \\ \text{(C)} & \psi = & \beta_n^\alpha + k_1\pi, & \theta = \psi + k_2\pi \\ \text{(D)} & \psi = & -\beta_n^\alpha + k_1\pi, & \theta = \psi + k_2\pi \end{array}$$

where $k_1, k_2 \in \mathbb{Z}$, and $\beta_n^\alpha := \arctan\left(\sqrt{\frac{\alpha}{n-1}}\right)$.

Lemma 9 *The system (12) has no nontrivial periodic solutions. Every trajectory in the ψ-θ -plane is bounded.*

Proof It is sufficient to consider $\theta \in \left(0, \frac{\pi}{2}\right)$. For every point in $(\psi, \theta) \in \left[\frac{\pi}{2}, \pi\right) \times \left(0, \frac{\pi}{2}\right)$, we have

$$\frac{d\psi}{dt} = \alpha \cos\psi \cos\theta - (n-1)\sin\psi \sin\theta \leq -(n-1)\sin\psi \sin\theta < 0$$

which means that the trajectories in this rectangle cannot be periodic and have to leave it at $\psi = \frac{\pi}{2}$. Once a trajectory is in $(\psi, \theta) \in \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right)$, it cannot leave this rectangle (Fig. 5). It remains to show that no periodic solutions are contained in this interval. Defining a function $B(\psi, \theta) := \sin^{\alpha-1}\theta$, we calculate

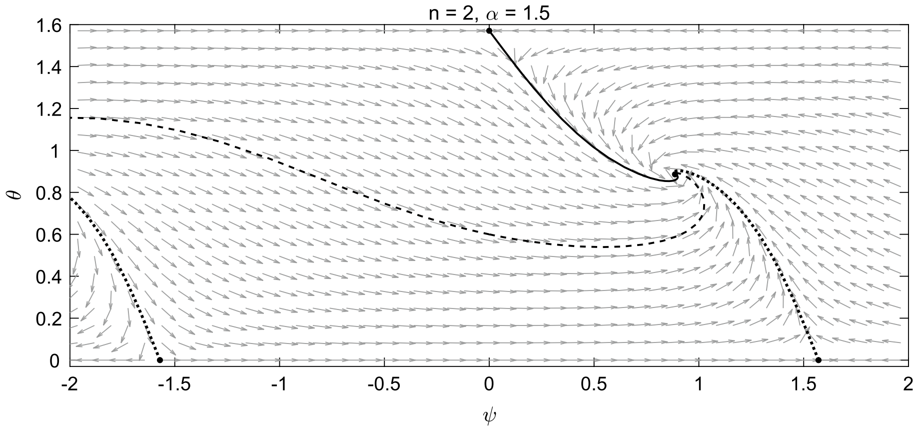

Fig. 5 ψ-θ-phase plane of (12) for $\alpha > 0$. Solutions of types (i), (iii) and (iv) are depicted by the solid, dotted and dashed curves, respectively. The n-α-cone (ii) is represented by the singular point $(\psi, \theta) = \left(\beta_n^\alpha, \beta_n^\alpha\right)$

$$
\begin{aligned}
&\frac{\mathrm{d}(Bf)}{\mathrm{d}\psi} + \frac{\mathrm{d}(Bg)}{\mathrm{d}\theta} \\
&= \sin^{\alpha-1}\theta(-\alpha\sin\psi\cos\theta - (n-1)\cos\psi\sin\theta \\
&\quad + \alpha\cos^2\theta\sin(\psi-\theta) - \sin^2\theta\sin(\psi-\theta) - \sin\theta\cos\theta\cos(\psi-\theta)\big) \\
&= \sin^{\alpha-1}\theta\big(-\alpha\sin\psi\cos\theta - (n-1)\cos\psi\sin\theta + \alpha\sin\psi\cos^3\theta \\
&\quad -\alpha\cos\psi\sin\theta\cos^2\theta - \sin\psi\sin^2\theta\cos\theta + \cos\psi\sin^3\theta - \sin\theta\cos\theta\cos(\psi-\theta)\big) \\
&\le \sin^{\alpha-1}\theta\big(-\alpha\sin\psi\sin^2\theta\cos\theta - \cos\psi\sin\theta\cos^2\theta \\
&\quad -\alpha\cos\psi\sin\theta\cos^2\theta - \sin\psi\sin^2\theta\cos\theta - \sin\theta\cos\theta\cos(\psi-\theta)\big) < 0.
\end{aligned}
$$

Using the Bendixson–Dulac theorem (see Perko [12, Section 3.9]) concludes the proof. □

Lemma 10 *Every trajectory of* (12) *within* $\theta \in \left(0, \frac{\pi}{2}\right)$ *approaches singular points as* $t \to \infty$ *and* $t \to -\infty$ *and - for each trajectory - at least one end point is of the form* $\left(\beta_n^\alpha + k\pi, \beta\alpha_n\right)$ *for some* $k \in \mathbb{Z}$. *In particular, every solution is of one of the four types* (*i*)–(*iv*) *in Theorem* 7, *and every case occurs.*

Proof By the Poincaré–Bendixson theorem (see Hartman [9, Section VII.4]) and Lemma 9, every trajectory approaches singular points at both end points. By virtue of the anti-periodicity of (12), it is sufficient to analyze the points $\left(\beta_n^\alpha, \beta_n^\alpha\right)$, $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, 0\right)$:

$\left(0, \frac{\pi}{2}\right)$: The matrix of the linearized system is

$$
\begin{pmatrix} 1-n & -\alpha \\ 0 & 1 \end{pmatrix}.
$$

The point is a saddle point, as in the case for $\alpha < 0$. The only trajectory in $\theta \in \left(0, \frac{\pi}{2}\right)$ starting at this point corresponds to the n-α-tectum (type (i)). It leaves the point with direction $(\alpha, -n)$, entering $(\psi, \theta) \in \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right)$ which was already shown in Lemma 8. It is impossible to leave this rectangle and as $t \to \infty$, the trajectory of the n-α-tectum must approach $\left(\beta_n^\alpha, \beta_n^\alpha\right)$.

$\underline{\left(\frac{\pi}{2}, 0\right)}$: Here we have the linearized system given by the matrix

$$\begin{pmatrix} -\alpha & 1-n \\ 0 & 1 \end{pmatrix}$$

and hence another saddle point. The only trajectory starting here leaves in the direction $(1-n, \alpha+1)$. Arguing as before we find that the trajectory has to approach the singular point $\left(\beta_n^\alpha, \beta_n^\alpha\right)$ and the solution curve is described by (iii).

$\underline{\left(\beta_n^\alpha, \beta_n^\alpha\right)}$: The equilibrium solution corresponds to the n-α-cone, cp. case (ii). The matrix of the linearized system is

$$A := \begin{pmatrix} -\sqrt{\alpha}\sqrt{n-1} & -\sqrt{\alpha}\sqrt{n-1} \\ \dfrac{\sqrt{\alpha}\sqrt{n-1}}{n-1+\alpha} & -\dfrac{\sqrt{\alpha}\sqrt{n-1}}{n-1+\alpha} \end{pmatrix}$$

with characteristic polynomial

$$\chi_A(\lambda) := \det(\lambda I - A) = \lambda^2 + \frac{(n+\alpha)\sqrt{\alpha}\sqrt{n-1}}{n-1+\alpha}\lambda + \frac{2\alpha(n-1)}{n-1+\alpha}.$$

The (complex) roots of χ_A are given by

$$\lambda_1 := \frac{\sqrt{\alpha}\sqrt{n-1}}{2(n-1+\alpha)}\left(-(n+\alpha) + \sqrt{(n+\alpha)^2 - 8(n+\alpha) + 8}\right),$$
$$\lambda_2 := \frac{\sqrt{\alpha}\sqrt{n-1}}{2(n-1+\alpha)}\left(-(n+\alpha) - \sqrt{(n+\alpha)^2 - 8(n+\alpha) + 8}\right).$$

The roots are real if and only if $n + \alpha \geq 4 + \sqrt{8}$. In any case both real parts are negative and we therefore have a stable node, if $n + \alpha \geq 4 + \sqrt{8}$, or a stable focus, if $n + \alpha < 4 + \sqrt{8}$.

All other trajectories in the phase space have to connect singular points of types $\left(\beta_n^\alpha + (2l-1)\pi, \beta_n^\alpha\right)$ and $\left(\beta_n^\alpha + 2k\pi, \beta_n^\alpha\right)$ for some $l, k \in \mathbb{Z}$. Additionally, since the phase plane is divided into regions of ψ-width π by the three other types of trajectories (i), (ii) and (iii), we must have $l \in \{k, k+1\}$. Hence, these trajectories correspond to solutions of system (9) which are described in case (iv) of Theorem 7. □

To conclude the proof of Theorem 7, it remains to analyze the asymptotic behavior of solutions.

Lemma 11 *Let $(x(s), y(s), \psi(s))$ be a maximal solution of (15) with $\lim_{s\to\infty} x(s) = \lim_{s\to\infty} y(s) = \infty$. Then $\gamma = (x, y)$ is asymptotic to the n-α-cone as $s \to \infty$. In other words, $\lim_{s\to\infty}\left(y(s) - \sqrt{\frac{\alpha}{n-1}}x(s)\right) = 0$.*

Proof Consider the following transformation of variables:

$$
\begin{aligned}
x &= \sqrt{\frac{n-1}{n-1+\alpha}}e^t(u-v) \qquad & u &= \frac{\sqrt{n-1+\alpha}}{2\sqrt{n-1}\sqrt{\alpha}s}\left(\sqrt{\alpha}x+\sqrt{n-1}y\right)\\
y &= \sqrt{\frac{\alpha}{n-1+\alpha}}e^t(u+v) \qquad & v &= \frac{\sqrt{n-1+\alpha}}{2\sqrt{n-1}\sqrt{\alpha}s}\left(\sqrt{n-1}y-\sqrt{\alpha}x\right)
\end{aligned}
$$

$$\Longleftrightarrow$$

$$
\begin{aligned}
\psi &= \phi+\beta_n^\alpha \qquad & \phi &= \psi-\beta_n^\alpha\\
s &= e^t & t &= \log s
\end{aligned}
$$

By virtue of Lemma 10, we infer $\lim_{s\to\infty}\phi = 0$ and, with de L'Hôpital's rule, also $\lim_{s\to\infty}u = 1$ and $\lim_{s\to\infty}v = 0$.

In these coordinates, $(u(t), v(t), \phi(t))$ solves the system:

$$
\begin{cases}
\dfrac{\mathrm{d}u}{\mathrm{d}t} = \cos\phi + \dfrac{n-1-\alpha}{2\sqrt{n-1}\sqrt{\alpha}}\sin\phi - u\\[2ex]
\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{n-1+\alpha}{2\sqrt{n-1}\sqrt{\alpha}}\sin\phi - v\\[2ex]
\dfrac{\mathrm{d}\phi}{\mathrm{d}t} = -\dfrac{\alpha}{u^2-v^2}\left(2\sqrt{\tfrac{n-1}{\alpha}}v\cos\phi + \left(\left(\tfrac{n-1}{\alpha}+1\right)u+\left(\tfrac{n-1}{\alpha}-1\right)v\right)\sin\phi\right).
\end{cases}
\tag{16}
$$

The linearized form of (16) near $(u, v, \phi) = (1, 0, 0)$ is

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}u\\ v\\ \phi\end{pmatrix} = \begin{pmatrix}-1 & 0 & \frac{n-1-\alpha}{2\sqrt{n-1}\sqrt{\alpha}}\\ 0 & -1 & \frac{n-1+\alpha}{2\sqrt{n-1}\sqrt{\alpha}}\\ 0 & -2\sqrt{n-1}\sqrt{\alpha} & -(n-1+\alpha)\end{pmatrix}\begin{pmatrix}u-1\\ v\\ \phi\end{pmatrix}
\tag{17}
$$

and the characteristic polynomial $\chi(\lambda) := (\lambda+1)\left(\lambda^2+(n+\alpha)\lambda+2(n+\alpha)-2\right)$ has the roots

$$
\lambda_1 := -1, \quad \lambda_2 := \frac{1}{2}\left(-(n+\alpha)+\sqrt{(n+\alpha)^2-8(n+\alpha)+8}\right),
$$
$$
\lambda_3 := \frac{1}{2}\left(-(n+\alpha)-\sqrt{(n+\alpha)^2-8(n+\alpha)+8}\right),
$$

where λ_2 and λ_3 are real if and only if $n+\alpha \geq 4+\sqrt{8}$. We always have $\mathrm{Re}(\lambda_2) < -1$ and $\mathrm{Re}(\lambda_3) < -1$.

Since λ_1 belongs to the eigenvector $e_1 = (1, 0, 0)$, we can estimate

$$
\left|e^t v\right| \leq \widetilde{c}_2 e^{(1+\mathrm{Re}(\lambda_2))t}\left|e^{i\mathrm{Im}(\lambda_2)t}\right| + \widetilde{c}_3 e^{(1+\mathrm{Re}(\lambda_3))t}\left|e^{i\mathrm{Im}(\lambda_3)t}\right| \to 0,
$$

with suitable constants $\widetilde{c}_2, \widetilde{c}_3 \in \mathbb{R}$ and as $s, t \to \infty$. This yields

$$\sqrt{n-1}y - \sqrt{\alpha}x = \frac{2\sqrt{n-1}\sqrt{\alpha}}{\sqrt{n-1+\alpha}}e^t v \to 0$$

again as $s, t \to \infty$. In other words, the n-α-cone is indeed an asymptotic line. □

Theorem 7 follows from Lemmata 8–11.

3.2 Intersections of the n-α-tectum and the n-α-cone

We show here that there are no points of intersection of the tectum and the cone, if for example $(n+\alpha) > 4 + \sqrt{8}$ and $n \geq 6$, cp. Theorem 14 and, for a more general result, see Theorem 15. In these cases the n-α-tectum turns out to be stable for symmetric perturbation, see Theorem 20. Recall that a point of intersection is characterized by the condition $\tan\theta = \sqrt{\frac{\alpha}{n-1}}$ and that the unstable manifold starting at the saddle point $(\psi, \theta) = (0, \frac{\pi}{2})$ has direction $(1, -\frac{n}{\alpha})$. Our aim is to construct a region $R \subset (0, \frac{\pi}{2}) \times (0, \frac{\pi}{2})$ in the ψ-θ-plane such that $R \subset \{\theta > \beta_n^\alpha\}$ and the vector field (12) points inward R whence solution trajectories of (12) remain trapped inside R and converge to the equilibrium $(\beta_n^\alpha, \beta_n^\alpha)$. To this end we let R denote the region enclosed by the three curves

$$\psi = 0 \qquad \text{(C1)}$$
$$\alpha = (n-1)\tan\psi\tan\theta \qquad \text{(C2)}$$
$$\tan\theta = a\tan\psi + b \qquad \text{(C3)}$$

where $a \leq 0$ (*so* $R \subset \{\theta > \beta_n^\alpha\}$) and $b := (1-a)\sqrt{\frac{\alpha}{n-1}}$, such that (C3) and (C2) intersect at $(\psi, \theta) = (\beta_n^\alpha, \beta_n^\alpha)$ (Fig. 6). Clearly, the vector field (12), when restricted to the curve (C1), points into R.

(C2) considered as the graph $\theta = g(\psi) = \arctan(\frac{\alpha}{(n-1)\tan\psi})$ has derivative $g'(\psi) = -\frac{\alpha}{n-1}\frac{\cos^2\theta}{\sin^2\psi} < 0$ and since - by equation (11) - every trajectory of (12) must have

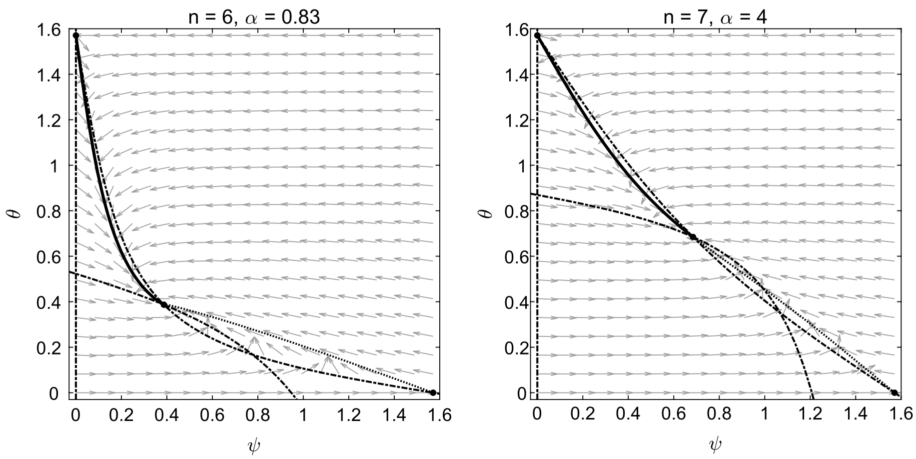

Fig. 6 The n-α-tectum contained in the region R enclosed by (C1)–(C3), represented by the dashed-dotted lines

$\frac{d\psi}{d\theta} = 0$ along (C2), it cannot leave R by intersecting (C2). Also note that $\lim_{\psi \to 0} g'(\psi) = \frac{1-n}{\alpha} > -\frac{n}{\alpha}$, whence the instable manifold (n-α-tectum) starting at $(0, \frac{\pi}{2})$ enters the region R.

Finally, we write (C3) as a graph $\theta = h(\psi) := \arctan(a \tan\psi + b)$ and find $h'(\psi) = a\frac{\cos^2\theta}{\cos^2\psi}$ and the condition $h(\psi) \leq g(\psi)$ requires that $a = h'(\beta_n^\alpha) \geq g'(\beta_n^\alpha) = -1$. By virtue of (11) any trajectory in R cannot leave R across curve (C3), if we can find some $a \in [-1, 0]$, such that for all ψ, θ along (C3) we have

$$\begin{aligned}
&a\frac{\cos^2\theta}{\cos^2\psi} < \sin\theta\cos\theta\frac{\tan\psi - \tan\theta}{\alpha - (n-1)\tan\psi\tan\theta} \\
\iff &a\big(\tan^2\psi + 1\big)(\alpha - (n-1)\tan\psi\tan\theta) < \tan\theta(\tan\psi - \tan\theta) \\
\iff &0 < a^2(n-1)\tan^4\psi + ab(n-1)\tan^3\psi + a(a(n-2) + 1 - \alpha)\tan^2\psi \\
&\qquad + ((n-3)a + 1)b\tan\psi - a\alpha - b^2.
\end{aligned}$$

In other words, defining the polynomial

$$\begin{aligned}
p_n^\alpha(a, z) = p_n^\alpha(z) :=\ &a^2(n-1)z^4 + ab(n-1)z^3 + a(a(n-2) + 1 - \alpha)z^2 \\
&+ ((n-3)a + 1)bz - a\alpha - b^2,
\end{aligned}$$

we have the following sufficient condition:

Lemma 12 *Suppose there exists an $a \in [-1, 0]$ such that*

$$p_n^\alpha(a, z) > 0 \quad \forall\, z \in \left(0, \sqrt{\frac{\alpha}{n-1}}\right). \tag{18}$$

Then the trajectory of the n-α-tectum stays inside the region R. In particular, it does neither intersect nor touch the n-α-cone.

Remark For $a < 0$, $p_n^\alpha(a, z)$ is a polynomial of degree four in z. Using $p_n^\alpha\left(a, \sqrt{\frac{\alpha}{n-1}}\right) = 0$, it can be reduced to a polynomial of degree three.

Theorem 13 *Let $n \geq 6$ and $\alpha > 0$ with $n + \alpha \geq 7$. Then the n-α-tectum does not intersect the n-α-cone.*

Proof Step 1: The choice of a.

We have to fulfill condition (18) of Lemma 12.

Setting $a := -\frac{1}{2}$ and transforming p_n^α for the sake of notational convenience we obtain

$$\begin{aligned}
q_n^\alpha(z) &:= 4\frac{n-1}{\alpha}p_n^\alpha\left(\sqrt{\frac{\alpha}{n-1}}z\right) \\
&= \alpha z^4 - 3\alpha z^3 + (n + 2\alpha - 4)z^2 + (15 - 3n)z + 2n - 11, \text{ with derivatives} \\
q_n^{\alpha\prime}(z) &= 4\alpha z^3 - 9\alpha z^2 + 2(n + 2\alpha - 4)z + 15 - 3n, \text{ and} \\
q_n^{\alpha\prime\prime}(z) &= 12\alpha z^2 - 18\alpha z + 2(n + 2\alpha - 4).
\end{aligned}$$

Now we have to show $q_n^\alpha(z) > 0$ on $(0, 1)$. Since $q_n^{\alpha\prime}(1) = 7 - (n + \alpha) \leq 0$, this is obviously fulfilled near the boundary points.

Step 2: Proof by monotonicity for small α.

$q_n^{\alpha}{}''$ has the roots $z = \frac{3}{4} \pm \sqrt{\frac{11}{48} + \frac{2}{3\alpha} - \frac{n}{6\alpha}}$ which are real if and only if

$$\frac{11}{48} + \frac{2}{3\alpha} - \frac{n}{6\alpha} \geq 0 \iff 11\alpha + 32 - 8n \geq 0 \iff \alpha \geq \frac{8}{11}n - \frac{32}{11}.$$

For $\alpha \leq \frac{8}{11}n - \frac{32}{11}$, $q_n^{\alpha}{}'$ is therefore increasing. Together with $q_n^{\alpha}{}'(1) \leq 0$, q_n^α must be decreasing on $(0, 1)$. This proves $q_n^\alpha(z) > 0$ for $\alpha \leq \frac{8}{11}n - \frac{32}{11}$.

Step 3: Proof by direct estimates for large α.

Now assume $\alpha > \frac{8}{11}n - \frac{32}{11}$. $q_n^{\alpha}{}'$ takes on a local maximum at $z_1 := \frac{3}{4} - \sqrt{\frac{11}{48} + \frac{2}{3\alpha} - \frac{n}{6\alpha}} \in \left(0, \frac{3}{4}\right)$. If $q_n^{\alpha}{}'\left(z_1\right) \leq 0$, we can again argue with monotonicity of q_n^α. Otherwise assume that $q_n^{\alpha}{}'\left(z_1\right) > 0$. Then $q_n^{\alpha}{}'$ has two roots in $(0, 1)$, of which we denote the first one by $z_2 \in \left(0, z_1\right)$. At this point, q_n^α attains its only local minimum and so we only have to show $q_n^\alpha\left(z_2\right) > 0$. Upon using $q_n^{\alpha}{}'\left(z_2\right) = 0$ and $z_2 < \frac{3}{4}$, we can estimate as follows:

$$\begin{aligned}
q_n^\alpha\left(z_2\right) &= q_n^\alpha\left(z_2\right) - \frac{1}{3}z_2 q_n^{\alpha}{}'\left(z_2\right)\\
&> \frac{1}{3}\left(n + \frac{23}{16}\alpha - 4\right)z_2^2 + (10 - 2n)z_2 + 2n - 11\\
&> \frac{1}{3}\left(2n - \frac{90}{11}\right)z_2^2 + (10 - 2n)z_2 + 2n - 11\\
&= \left(\frac{2}{3}n - \frac{30}{11}\right)\left(z_2 + \frac{5-n}{\frac{2}{3}n - \frac{30}{11}}\right)^2 + 2n - 11 - \frac{(5-n)^2}{\frac{2}{3}n - \frac{30}{11}}\\
&\geq \frac{(2n-11)\left(\frac{2}{3}n - \frac{30}{11}\right) - (5-n)^2}{\frac{2}{3}n - \frac{30}{11}} = \frac{\frac{1}{3}n^2 - \frac{92}{33}n + 5}{\frac{2}{3}n - \frac{30}{11}} > 0
\end{aligned}$$

We conclude that (18) is satisfied and by Lemma 12 also Theorem 13 follows. □

Remark Theorem 13 is optimal for $n \geq 7$ and nearly optimal for $n = 6$. There is no result for $n \leq 5$, and we can in fact see that Lemma 12 requires $n \geq 6$ as follows by looking at $p_n^\alpha(0)$:

$$\begin{aligned}
0 \leq p_n^\alpha(0) &= -a\alpha - b^2 = -\left(a^2 + (n-3)a + 1\right)\frac{\alpha}{n-1}\\
\iff 0 \geq a^2 + (n-3)a + 1 &= \left(a + \frac{n-3}{2}\right)^2 + 1 - \frac{(n-3)^2}{4}.
\end{aligned}$$

This polynomial has roots $-\frac{n-3}{2} \pm \frac{\sqrt{(n-1)(n-5)}}{2}$, which implies that $n \geq 5$. Also, plugging in $n = 5$ and $a := -\frac{n-3}{2} = -1$ results in $p_n^{\alpha}{}'(0) = ((n-3)a + 1)b = -\sqrt{\alpha} < 0$.

So the only way of improving Theorem 13 with the same region R is by weakening the requirement $n + \alpha \geq 7$. Recall that, this was needed for $p_n^{\alpha}{}'\left(\sqrt{\frac{\alpha}{n-1}}\right) \leq 0$. Let us now study this inequality for different values of $a \in [-1, 0]$. Writing $p_n^{\alpha}{}'(a, z)$ as

$$p_n^{\alpha}{}'(z) = 4a^2(n-1)z^3 + 3ab(n-1)z^2 + 2a(a(n-2) + 1 - \alpha)z + ((n-3)a + 1)b$$

and plugging in the boundary point yields

$$p_n^{\alpha\prime}\Big(\sqrt{\tfrac{\alpha}{n-1}}\Big) = \sqrt{\tfrac{\alpha}{n-1}}\big((n+\alpha-1)a^2 + (n+\alpha-2)a + 1\big).$$

Viewing this as a polynomial in a, the roots depend only on the sum of n and α and are real if and only if

$$\begin{aligned}\frac{1}{4}(n+\alpha-2)^2 - (n+\alpha-1) \geq 0 \iff& (n+\alpha)^2 - 8(n+\alpha) + 8 \geq 0\\ \iff& n+\alpha \geq 4+\sqrt{8},\end{aligned}$$

a value which looks familiar from the proofs of Theorems 7 and Lemma 11.

Using this observation, we will now improve Theorem 13.

Theorem 14 *Let $n \geq 6$ and $\alpha > 0$ with $n+\alpha > 4+\sqrt{8}$. Then the n-α-tectum does not intersect the n-α-cone.*

Proof In order to minimize $p_n^{\alpha\prime}\Big(\sqrt{\tfrac{\alpha}{n-1}}\Big)$, we put $a := -\frac{1}{2}\cdot\frac{n+\alpha-2}{n+\alpha-1}$. Then set

$$\begin{aligned}r_n^\alpha(z) :=& \frac{(n-1)(n+\alpha-1)^2}{\alpha(z-1)} p_n^\alpha\left(\sqrt{\frac{\alpha}{n-1}}z\right)\\ =& \Big(\tfrac{1}{4}(n+\alpha)^2 - (n+\alpha) + 1\Big)\alpha z^3 + \Big(-\tfrac{1}{2}(n+\alpha)^2 + \tfrac{3}{2}(n+\alpha) - 1\Big)\alpha z^2\\ &+ \Big(\Big(\tfrac{1}{4}(n+\alpha)^2 - (n+\alpha) + 1\Big)n - (n+\alpha)^2 + \tfrac{7}{2}(n+\alpha) - 3\Big)z\\ &+ \Big(-\tfrac{1}{2}(n+\alpha)^2 + \tfrac{3}{2}(n+\alpha) - 1\Big)n + \tfrac{11}{4}(n+\alpha)^2 - \tfrac{15}{2}(n+\alpha) + 5.\end{aligned}$$

We have to show that $r_n^\alpha(z) < 0$ for all $z \in (0,1)$ in order to be able to apply Lemma 12. From our choice of a and $n+\alpha > 4+\sqrt{8}$, we already know this is fulfilled near the boundary.

Without loss of generality, let us assume $n = 6$. Then $r_6^\alpha(z)$ becomes

$$\begin{aligned}&\Big(\tfrac{1}{4}\alpha^2+2\alpha+4\Big)\alpha z^3 - \Big(\tfrac{1}{2}\alpha^2+\tfrac{9}{2}\alpha+10\Big)\alpha z^2 + \Big(\tfrac{1}{2}\alpha^2+\tfrac{7}{2}\alpha+6\Big)z - \tfrac{1}{4}\alpha^2 - \tfrac{3}{2}\alpha - 1\\ &= \tfrac{1}{4}(\alpha+4)^2\alpha z^3 - \tfrac{1}{2}(\alpha+5)(\alpha+4)\alpha z^2 + \tfrac{1}{2}(\alpha+4)(\alpha+3)z - \tfrac{1}{4}\alpha^2 - \tfrac{3}{2}\alpha - 1.\end{aligned}$$

We observe $r_6^\alpha(0) < -1 < 0$, and the roots of the derivative of r

$$r_6^{\alpha\prime}(z) = \tfrac{3}{4}(\alpha+4)^2\alpha z^2 - (\alpha+5)(\alpha+4)\alpha z + \tfrac{1}{2}(\alpha+4)(\alpha+3)$$

are $\frac{2}{3}\cdot\frac{\alpha+5}{\alpha+4} \pm \sqrt{\Big(\frac{2}{3}\cdot\frac{\alpha+5}{\alpha+4}\Big)^2 - \frac{2}{3}\cdot\frac{\alpha+3}{(\alpha+4)\alpha}}$. Since $\alpha \geq \sqrt{8}-2$, these are real and distinct. Hence, r_6^α attains a local maximum at

$$z_1(\alpha) = z_1 := \frac{2}{3}\cdot\frac{\alpha+5}{\alpha+4} - \sqrt{\left(\frac{2}{3}\cdot\frac{\alpha+5}{\alpha+4}\right)^2 - \frac{2}{3}\cdot\frac{\alpha+3}{(\alpha+4)\alpha}} \in (0,1)$$

and it remains to be shown that $r_6^\alpha\big(z_1(\alpha)\big) < 0$. For this, consider

$$r_6^\alpha\left(z_1(\alpha)\right) = r_6^\alpha\left(z_1\right) - \tfrac{1}{3}z_1 r_6^{\alpha\prime}\left(z_1\right) = q_{z_1}(\alpha) \tag{19}$$

where $q_z(\alpha)$ is defined as the following function in α:

$$\begin{aligned} q_z(\alpha) &:= -\tfrac{1}{6}(\alpha+5)(\alpha+4)\alpha z^2 + \tfrac{1}{3}(\alpha+4)(\alpha+3)z - \tfrac{1}{4}\alpha^2 - \tfrac{3}{2}\alpha - 1. \\ q_z'(\alpha) &= -\left(\tfrac{1}{2}\alpha^2 + 3\alpha + \tfrac{10}{3}\right)z^2 + \left(\tfrac{2}{3}\alpha + \tfrac{7}{3}\right)z - \tfrac{1}{2}\alpha - \tfrac{3}{2} \\ q_z''(\alpha) &= -(\alpha+3)z^2 + \tfrac{2}{3}z - \tfrac{1}{2} \end{aligned}$$

We have $q_z''(\alpha) < 0$, so

$$q_z'(\alpha) < q_z'\left(\sqrt{8}-2\right) = \left(-\tfrac{10}{3} - 2\sqrt{2}\right)z^2 + \left(1 + \tfrac{4}{3}\sqrt{2}\right)z - \tfrac{1}{2} - \sqrt{2} < 0.$$

In the same way, it follows that

$$q_z(\alpha) < q_z\left(\sqrt{8}-2\right) = \left(-2 - \tfrac{4}{3}\sqrt{2}\right)\left(z - \tfrac{1}{2}\sqrt{2}\right)\left(z - \left(3 - \tfrac{3}{2}\sqrt{2}\right)\right)$$

which means that $q_z(\alpha) < 0$ if $z \in \left(0, \frac{1}{\sqrt{2}}\right)$. By (19), all we have to show is $z_1(\alpha) < \frac{1}{\sqrt{2}}$. This follows from $z_1\left(\sqrt{8}-2\right) = \frac{1}{\sqrt{2}}$ and $z_1'(\alpha) < 0$ for all $\alpha \geq \sqrt{8}-2$. □

Remark The planar system (12) is—modulo obvious transformations—equivalent to the ordinary differential system (1) or (13) in [4] (see also [3]), which has been thoroughly investigated. According to Lemma 1 in [4] there exist trajectories γ_1 starting at the singular point $(2\pi, 0)$ and ending at $(\pi, 0)$ provided one of the following conditions holds:

(i) $\alpha \geq 3$, $p := (n-1) \geq 3$
(ii) $p \geq 2$, $\alpha \geq 4$, or
(iii) $p \geq 1$, $\alpha \geq 6$

which complement the sufficient conditions

(ii′) $\alpha \geq 2$, $p \geq 4$ or
(iii′) $\alpha \geq 1$, $p \geq 6$

of Lemma 2 in [4].

Using Lemmata 1 & 2 in [4] we obtain the following result, which is supplementary to Theorem 14:

Theorem 15 *Suppose that one of the conditions* (*i*), (*ii*), (*ii′*) *or* (*iii*) *holds. Then the n-α-tectum does not intersect the n-α-cone.*

Note that there is still room for improving these conditions, similarly as in Theorem 14. The following symmetry property of system (12)—which can easily be verified—might be useful in this respect: Suppose the trajectory $(\psi_1(t), \theta_1(t))$, $t \in \mathbb{R}$, of system (12) starts at $(0, \frac{\pi}{2})$ and ends at $(\beta_n^\alpha, \beta_n^\alpha)$, where $\beta_n^\alpha := \arctan\sqrt{\frac{\alpha}{p}}$, $p = (n-1)$, then the curve $(\psi_0(t), \theta_0(t)) := (\frac{\pi}{2} - \psi_1(t), \frac{\pi}{2} - \theta_1(t))$ solves again system (12) with α replaced by p and p replaced by α, starts at $(\frac{\pi}{2}, 0)$ and ends at the equilibrium point $(\overline{\beta}_n^{\alpha}, \overline{\beta}_n^{\alpha})$ where $\overline{\beta}_n^{\alpha} := \arctan\sqrt{\frac{p}{\alpha}}$, $p = n-1$.

3.3 Stability and minimizing properties of the *n*-α-tectum

The energy integral

$$E_\alpha(u) := \int_\Omega u^\alpha \sqrt{1+|Du|^2}\,\mathrm{d}x$$

for rotational symmetric surfaces $M = graph(u)$ generated by curves $u = y(r), r = |x|$, is proportional to the variational integral

$$I(u) := \int_a^b r^{n-1} u(r)^\alpha \sqrt{1+u'(r)^2}\,\mathrm{d}r \tag{20}$$

whose Euler equation is given in (5). We say that the n-α-tectum $y(\cdot)$ is *symmetrically stable*, if the second variation $\delta^2 I(y,\phi) > 0$ for all $\phi \in C_c^1(a,b)$. Also recall that for the general one-dimensional variational integral $\int_a^b f(x,y,y')\,\mathrm{d}x$ we have Jacobi's equation

$$\left(P - \frac{\mathrm{d}Q}{\mathrm{d}x}\right)h - \frac{\mathrm{d}}{\mathrm{d}x}\left(R\frac{\mathrm{d}h}{\mathrm{d}x}\right) = 0,$$

where $P := f_{zz}(\cdot,y,y'), Q := f_{zp}(\cdot,y,y')$ and $R := f_{pp}(\cdot,y,y')$ see, e.g., Bolza [2, chap.2,§9-§12].

Here we obtain Jacobi's equation simply as follows

$$\begin{aligned}\alpha\left[(\alpha-1)x^{n-1}y^{\alpha-2}\sqrt{1+y'^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{n-1}y^{\alpha-1}y'}{\sqrt{1+y'^2}}\right)\right]h &= \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{n-1}y^\alpha}{(1+y'^2)^{\frac{3}{2}}}h'\right), &\qquad \text{where } x > 0.\end{aligned} \tag{21}$$

We have the following

Proposition 16 *Let $y = y(x)$ stand for the n-α-tectum. Then the function $h(x) := y(x) - xy'(x)$ solves Jacobi's equation (21) on all of $\mathbb{R}$. Furthermore, h is unique up to constant multiples.*

Proof h is symmetric, since y is symmetric and from (15) we infer

$$y'' = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)' = \frac{\mathrm{d}}{\mathrm{d}s}(\tan\psi)\frac{\mathrm{d}s}{\mathrm{d}x} = \left(1+y'^2\right)\left(\frac{\alpha}{y} - (n-1)\frac{y'}{x}\right)$$

whence (21) follows by direct computations. Uniqueness is a consequence of the requirement $h'(0) = 0$. □

Theorem 17 *Suppose that $\theta > \psi$ on some interval $[0, x_1]$ of the n-α-tectum y. Then y is symmetrically stable on $[-x_1, x_1]$. On the other hand, if at some point in the interval $[0, x_1]$ we have $\theta = \psi$, then there is some nontrivial variation ϕ such that $\delta^2 I(y, \phi) = 0$.*

Proof If $\theta > \phi$ on $[0, x_1]$ the function $h = y - xy'$ has no zeros on $[0, x_1]$ or $[-x_1, x_1]$ and hence there are no conjugate points on $[0, x_1]$ by Proposition 16. Otherwise, $[-x_1, x_1]$ contains a pair of conjugate points and by Jacobi's theory (see, e.g., Bolza [2, §10, 12]) there is a nontrivial function h such that $\delta^2 I(y, h) = 0$. □

Corollary 18 *Let $\alpha > 0$ and $n + \alpha < 4 + \sqrt{8}$, then every n-α-tectum is not (globally) stable.*

Proof The singular points $(\beta_n^\alpha, \beta_n^\alpha)$ are stable foci provided $n + \alpha < 4 + \sqrt{8}$, see the discussion in the proof of Lemma 10, whence the Corollary follows from Theorem 17. □

In particular we have the following Bernstein-type result for solutions of the s.m.s.e. (1) if $\alpha > 0$.

Theorem 19 *Let $\alpha > 0$ and $n + \alpha < 4 + \sqrt{8}$. Then there is no stable, entire rotationally symmetric solution $u \in C^2(\mathbb{R}^n)$ of the s.m.s.e.(1).*

The analogue result for *not necessarily symmetric, stable entire C^2-solutions* of equation (1) was shown to hold under the stronger condition $(n + \alpha) < 4 + \sqrt{\frac{2}{n+\alpha}}$, i.e., $(n + \alpha) < 5.23...$, cp. Dierkes [6]. We conjecture that Theorem 19 holds for all stable, entire solutions of equation (1). The assertion of Theorem 19 also holds for Lipschitz regular cones with vertex at the origin, see the paper [5].

We conclude with the following result on stability and minimizing properties of the *n-α* -tecti, if $n + \alpha > 4 + \sqrt{8}$ *and* $\alpha > 0$.

Theorem 20

(A) *Let $n \geq 6$ and $n + \alpha > 4 + \sqrt{8}$, then we have*

- (i) *The n-α-tectum is (symmetrically) stable.*
- (ii) *The n-α-tectum locally minimizes the energy E_α with respect to variations in the set $A := \{x \in \mathbb{R}^n \times \mathbb{R}^+; x_{n+1} > \sqrt{\frac{\alpha}{n-1}}(x_2^1 + ... + x_n^2)^{\frac{1}{2}}\}$.*
- (iii) *The* 6-1-*tectum, although stable and locally minimizing, does not minimize the energy $E_1(\cdot)$ with respect to variations in the upper half space $\mathbb{R}^6 \times \mathbb{R}^+$.*

(B) *Suppose that one of the following conditions holds*:

- (i) $\alpha,\ p \geq 3$, *where* $p := (n - 1)$,
- (ii) $p \geq 2,\ \alpha \geq 4$ *or* $\alpha \geq 2, p \geq 4$
- (iii) $p \geq 1,\ \alpha \geq 6$ *or* $\alpha \geq 1,\ p \geq 6$.

In all these cases the n-α-tectum minimizes the energy E_α in suitable classes of nonnegative functions with bounded variation.

Proof ad A): By Theorem 14 we have $\tan\theta > \sqrt{\frac{\alpha}{n-1}} > \tan\psi$, so Theorem 17 implies stability of the *n*-α-tectum; hence, (i) follows. Furthermore, from Lemma 12 and

Theorem 14 we obtain the existence of a *field* (or *"calibration"*) of n-α-tecti lying completely above the n-α-cone. This in turn leads to a function of least α-gradient which is defined in the open set $A := \{x \in \mathbb{R}^n \times \mathbb{R}^+; x_{n+1} > \sqrt{\frac{\alpha}{n-1}}(x_1^2 + ... + x_n^2)^{\frac{1}{2}}\}$; since the procedure is analogue to the construction device in [4] and [3] we skip over the details, referring to the papers [3, 4]. It follows that every n-α-tectum minimizes (in a very general sense) the energy E_α with respect to variations inside the set A. For a more detailed discussion we refer to the papers [3, 4]

ad (iii): If the 6-1-tectum were minimizing, another minimizing solution v could be constructed which were not of class C^1 in the set $\{x \in \mathbb{R}^n; v(x) > 0\}$, which, however, contradicted known regularity results proved in [1]. Since the argument is similar to the proof of Theorem 2 in [4], we omit the details and refer the reader to [4].

ad B): Under the conditions (i),(ii) or (iii) the minimizing property of the n-α-tecti was implicitly established in the proof of Theorem 1 of [4], c.p. also [3].

The proof follows from the construction of the function f with least α-gradient in the upper half space $\mathbb{R}^n \times \mathbb{R}^+$ which has the n-α-cones and the n-α-tecti as level surfaces. Lemmata 1 & 2 of [4] guarantee the existence of the field (or "calibration") which leads to the function f, while in [3] the minimizing properties of the corresponding level sets are proved. Since the n-α-tecti appear as level surfaces of the function f in the respective cases, they are minimizers of the energy integral $E_\alpha(\cdot)$. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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