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Research Article

Multiplicity Solutions of Fractional Impulsive p-Laplacian Systems: New Result

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In this paper, the existence of multiplicity distinct weak solutions is proved for differentiable functionals for perturbed systems of impulsive nonlinear fractional differential equations. Further, examples are given to show the feasibility and efficacy of the key findings. This work is an extension of the previous works to Banach space.

1. Introduction

This paper explores the perturbed impulsive fractional differential system

$$\begin{cases} {}_tD_T^{\alpha_i}\left(\phi_p\left({}_0D_t^{\alpha_i}u_i(t)\right)\right) = \lambda F_{u_i}(t,u) + \mu G_{u_i}(t,u) + h_i(u_i(t)), & t \in [0,T], \\ \Delta\left({}_tD_T^{\alpha_i-1}\phi_p\left({}_0^cD_t^{\alpha_i}u_i(t)\right)(t_j)\right) = I_{ij}(u_i(t_j)), & 1 \le i \le n,\ j = 1, 2, \cdots, m, \\ u_i(0) = u_i(T) = 0, \end{cases} \tag{1}$$

where $u = (u_1, u_2, \cdots, u_n)$, $n \ge 1$, $0 < \alpha_i \le 1$ for $1 \le i \le n$, ${}_0D_t^{\alpha_i}$ and ${}_tD_T^{\alpha_i}$ are the left and right Riemann–Liouville fractional derivatives of order α_i, respectively, $\varphi_p(s) = |s|^{p-2}s$, $s \ne 0$, $\varphi_p(0) = 0$, $p > 1$, $\lambda > 0$, $\mu > 0$, $T > 0$, and $F, G : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are L^1-Caratheodory functions, and they satisfy in the following standard summability condition:

$$\sup_{|\xi| \le \varsigma_1}\left(\max\left\{|F(.,\xi)|, |G(.,\xi)|, \left|F_{\xi i}(.,\xi)\right|, \left|G_{\xi i}(.,\xi)\right|, i = 1, \cdots, n\right\}\right) \in L^1([0,T]), \tag{2}$$

for any $\varsigma_1 > 0$ with $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ and $|\xi| = \sqrt[p]{\sum_{i=1}^n \xi_i^p}$, and $h_i : \mathbb{R} \longrightarrow \mathbb{R}$ is a $(p-1)$-Lipschitz continuous function with the Lipschitz constant $L_i > 0$, i.e.,

$$|h_i(\xi_1) - h_i(\xi_2)| \le L_i|\xi_1 - \xi_2|^{p-1}, \tag{3}$$

for every $\xi_1, \xi_2 \in \mathbb{R}$, satisfying $h_i(0) = 0$ for $1 \le i \le n$. The operator Δ is defined as $0 < t_0 < t_1 < \cdots < t_m + 1 = T$ and

$$\Delta\left({}_tD_T^{\alpha_{i-1}}\phi_p\left({}_0^cD_t^{\alpha_i}u_i\right)(t_j)\right) = {}_tD_T^{\alpha_{i-1}}\phi_p\left({}_0^cD_t^{\alpha_i}u_i\right)\left(t_j^+\right) - {}_tD_T^{\alpha_{i-1}}\phi_p\left({}_0^cD_t^{\alpha_i}u_i\right)\left(t_j^-\right), \tag{4}$$

where

$$
\begin{aligned}
&{}_tD_T^{\alpha_{i-1}}\phi_p\left({}_0^cD_t^{\alpha_i}u_i\right)\left(t_j^+\right)=\lim_{t\longrightarrow t_j^+}{}_tD_T^{\alpha_{i-1}}\phi_p\left({}_0^cD_t^{\alpha_i}u_i\right)(t),\\
&{}_tD_T^{\alpha_{i-1}}\phi_p\left({}_0^cD_t^{\alpha_i}u_i\right)\left(t_j^-\right)=\lim_{t\longrightarrow t_j^-}{}_tD_T^{\alpha_{i-1}}\phi_p\left({}_0^cD_t^{\alpha_i}u_i\right)(t),
\end{aligned}
\tag{5}
$$

and $I_j\in C(\mathbb{R},\mathbb{R})$ is a $(p-1)$-Lipschitz continuous function with the Lipschitz constant $L_{ij}>0$, i.e.,

$$\left|I_j(\xi_1)-I_j(\xi_2)\right|\le L_{ij}|\xi_1-\xi_2|^{p-1}. \tag{6}$$

Here, F_{u_i} and G_{u_i} are the partial derivatives of F and G with respect to u_i for $1\le i\le n$, respectively.

In science and engineering, fractional differential equations (FDEs) have recently proved to be useful methods for modeling a broad variety of phenomena. In viscoelasticity, electrochemistry, power, porous media, and electromagnetism, for instance, see [1–33] and the references therein. Many articles have recently investigated the existence of solutions to boundary value problems for FDEs, and we refer the reader to one of them [2, 18–20, 34–46] and the references therein. For example, Kamache et al. [40] investigated the existence of three solutions for a class of fractional p-Laplacian systems using a variational structure and critical point theory.

In [36], we investigated the existence of solutions of the periodic boundary value problem for a nonlinear impulsive fractional differential equation with periodic boundary conditions:

$$
\begin{aligned}
D^{2\alpha}u(t)&=f(t,u,Du),\\
&\quad t\in(0,1]\setminus t_0,t_1,\cdots,t_m,\ 0<\alpha<1,
\end{aligned}
\tag{7}
$$

$$\lim_{t\longrightarrow 0+}t^{1-\alpha}u(t)=u(1), \tag{8}$$

$$\lim_{t\longrightarrow 0+}t^{1-\alpha}D^{\alpha}u(t)=D^{\alpha}u(1), \tag{9}$$

$$\lim_{t\longrightarrow t_j^+}\left(t-t_j\right)^{1-\alpha}\left(u(t)-u\left(t_j\right)\right)=I_j(u(t)), \tag{10}$$

$$\lim_{t\longrightarrow t_j^+}\left(t-t_j\right)^{1-\alpha}\left(D^{\alpha}u(t)-D^{\alpha}u\left(t_j\right)\right)=\bar{I}_j(u(t)), \tag{11}$$

where $D^{\alpha}u(t)=({}_0D_t^{\alpha}u)(t)=(1/(\Gamma(2-\alpha)))(d/dt)\int_0^t(t-\tau)^{-\alpha}u(\tau)d\tau$ is the standard Riemann–Liouville fractional derivative, $D^{2\alpha}u=D^{\alpha}(D^{\alpha}u)$ is the sequential Riemann–Liouville fractional derivative presented by Miller and Ross on p. 209 of [14], $0<t_0<t_1<\cdots<t_m=1$, $I_j,\bar{I}_j\in C(\mathbb{R},\mathbb{R})(j=1,\cdots,m)$, and f is continuous at every point $(t,u,v)\in[0,1]\times\mathbb{R}\times\mathbb{R}$. By using the method of upper and lower solutions and its associated monotone iterative method, the author studies the existence and uniqueness of the solution of the periodic boundary value problem for the nonlinear impulsive fractional differential equation (7).

Upon using variational methods and critical point theory, the presence of one weak solution for the system was also demonstrated in [19] with $\mu=0$ and $I_{ij}=0$ for $i=1,\cdots,n$ and $j=1,\cdots,m$.

Impulsive effects are a common phenomenon triggered by short-term perturbations that are negligible in relation to the original operation's total duration. Such perturbations can be approximated fairly well as instantaneous changes of state or in the form of impulses. Such phenomena governing equations can be interpreted as impulsive differential equations. There has been a surge in interest in the study of impulsive differential equations in recent years, as these equations provide a natural framework for mathematical modeling of many real-world phenomena, especially in control theory, physics, chemistry, population dynamics, biotechnology, economics, and medical fields. Under such boundary conditions, the presence of solutions for impulsive differential equations with variational structures is determined by variational methods. See, for example, [36] as well as the references therein. Many scholars have recently studied fractional differential equations with impulses using variational methods, fixed point theorems, and critical point theory, due to the rapid growth in the theory of fractional calculus and impulsive differential equations, as well as their broad applications in a variety of fields (see, for example, [35, 44] and the references therein for a thorough discussion, as well as the sources therein for more details). For example, Gao et al. provided sufficient conditions for the existence and uniqueness of solutions for a class of impulsive integrodifferential equations with nonlocal conditions involving the Caputo fractional derivative using the Schaefer fixed point theorems (see [45]).

The existence of infinitely many solutions for the system (1) was discussed in [46] using variational methods. Some new parameters to guarantee that the system (1), in the case $\mu=0$, has at least two nontrivial and nonnegative solutions were obtained in [30] under appropriate hypotheses and using variational methods.

Recently, in Reference [27], perturbed systems of impulsive nonlinear fractional differential equations were studied, including continuous nonlinear Lipschitz terminology where at least three distinct weak solutions were demonstrated based on the modern critical point theory of differentiable functions, but here, we will prove the existence of three distinct weak solutions for differentiable functionals for perturbed systems of impulsive nonlinear fractional differential equations.

Most precisely, in this work, we extend the last work [38] to Banach space, where we show that there are at least three weak solutions for the system (1), which involves two parameters λ and μ. Furthermore, we do not need any asymptotic conditions of the nonlinear term at infinity in our new findings. The proof is based on a three-critical point theorem proved by Bonanno and Candito in [32], which we will revisit in the following section (Theorem 1). Theorem 10 is our most important finding. As a result, Theorem 11 can be deduced. Theorem 11 is shown in Example 1. When it comes to a scalar situation ($n=1$), we obtain Theorems 14 and 15 as special cases of Theorems 10 and 11. Theorem 15 is shown in Example 2. Under appropriate conditions on the nonlinear term at zero and at infinity, we obtain the presence of at least two positive solutions in Theorem 16.

The present paper is organized as follows. In Section 2, we recall some basic definitions and preliminary results, while Section 3 is devoted to the existence of multiple weak solutions for the eigenvalue system (1).

2. Preliminaries

Let X be a nonempty set and $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two functions. For all $r_1, r_2, r_3 > \inf_X \Phi$, $r_2 > r_1, r_3 > 0$, we define

$$\begin{aligned} \varphi(r) &:= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}, \\ \beta(r_1, r_2) &:= \inf_{u\in\Phi^{-1}(-\infty,r)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)}, \\ \gamma(r_2, r_3) &:= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}}{r_3}, \\ \alpha(r_1, r_2, r_3) &:= \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}. \end{aligned} \tag{12}$$

Theorem 1 ([32], Theorem 3.3). *Let X be a reflexive real Banach space; let $\Phi : X \longrightarrow \mathbb{R}$ be a coercive and continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on X^*, where X^* is the dual space of X, and let $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gateaux differentiable functional whose Gateaux derivative is compact, such that*

(a_1): Φ is convex, and $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.

(a_2): for every $u_1, u_2 \in X$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has

$$\inf_{s\in[0,1]} \Psi(su_1 + (1 - s)u_2) \geq 0. \tag{13}$$

Assume that there are three positive constants r_1, r_2, and r_3 with $r_1 < r_2$, such that

$$\begin{aligned} &(a_3)\varphi(r_1) < \beta(r_1, r_2), \\ &(a_4)\varphi(r_2) < \beta(r_1, r_2), \\ &(a_5)\gamma(r_2, r_3) < \beta(r_1, r_2). \end{aligned} \tag{14}$$

Then, for each $\lambda \in]1/\beta(r_1, r_2), 1/\alpha(r_1, r_2, r_3)[$, the functional $\Phi - \lambda\Psi$ admits three distinct critical points u_1, u_2, and u_3 such that $u_1 \in \Phi^{-1}(-\infty, r_1)$, $u_2 \in \Phi^{-1}[r_1, r_2)$, and $u_3 \in \Phi^{-1}(-\infty, r_2 + r_3)$.

Now, we introduce some important fractional calculus concepts and properties that will be used in this paper.

Let $C_0^\infty([0, T], \mathbb{R}^n)$ be the set of all functions $x \in C_0^\infty([0, T], \mathbb{R}^n)$ with $x(0) = x(T) = 0$ and the norm

$$\|x\|_\infty = \max_{t\in[0,T]} |x(t)| \tag{15}$$

Denote the norm of the space $L^p([0, T], \mathbb{R}^n)$ for $1 \leq p < \infty$ by

$$\|x\|_{L^p} = \int_0^T |x(s)|^p ds. \tag{16}$$

The following lemma yields the boundedness of the Riemann–Liouville fractional integral operators from the space $L^p([0, T], \mathbb{R}^n)$ to the space $L^p([0, T], \mathbb{R}^n)$, where $1 \leq p < \infty$.

Definition 2 [35]. The left and right Riemann–Liouville fractional derivatives of order α_i for the function u are defined in the following forms, respectively,

$$\begin{aligned} {}_0D_t^{\alpha_i} u(t) &= \frac{1}{\Gamma(1 - \alpha_i)} \frac{d}{dt} \left(\int_0^t (t - s)^{-\alpha_i} u(s) ds \right), \quad t > 0, \\ {}_tD_t^{\alpha_i} u(t) &= \frac{1}{\Gamma(1 - \alpha_i)} \frac{d}{dt} \left(\int_t^T (s - t)^{-\alpha_i} u(s) ds \right), \quad t < T, \end{aligned} \tag{17}$$

where u is a function defined on $[0, T]$ and $\alpha_i > 0$ for $1 \leq i \leq n$, and $\Gamma(\alpha_i)$ is the standard gamma function given by

$$\Gamma(z) = \int_0^{+\infty} z^{\alpha_i - 1} e^{-z} dz. \tag{18}$$

Definition 3 (see [40]). Let $\alpha_i \geq 0$ for $1 \leq i \leq n$ and $n \in \mathbb{N}$.

(i) If $\alpha_i \in (n - 1, n)$ and $u \in AC^n([0, T], \mathbb{R}^n)$, then the left and right Caputo fractional derivatives of order α_i for function u denoted by ${}_0^cD_t^{\alpha_i}u$ and ${}_t^cD_T^{\alpha_i}u$, respectively, exist almost everywhere on $[0, T]$ and for $1 \leq i \leq n$, where ${}_0^cD_t^{\alpha_i}u$ and ${}_t^cD_T^{\alpha_i}u$ are represented by

$$\begin{aligned} {}_0^cD_t^{\alpha_i} u &= \frac{1}{\Gamma(n - \alpha_i)} \int_0^t (t - s)^{n-\alpha_i-1} u^{(n)}(s) ds, \quad t \in [0, T], \\ {}_t^cD_T^{\alpha_i} u &= \frac{(-1)^n}{\Gamma(n - \alpha_i)} \int_t^T (t - s)^{n-\alpha_i-1} u^{(n)}(s) ds, \quad t \in [0, T], \end{aligned} \tag{19}$$

respectively.

(ii) If $\alpha_i = n - 1$ and $u \in AC^n([0, T], \mathbb{R}^n)$, then ${}_0^cD_t^{n-1}u(t)$ and ${}_t^cD_T^{n-1}u(t)$ are represented by ${}_0^cD_t^{n-1}u(t) = u^{(n-1)}(t)$ and ${}_t^cD_T^{n-1}u(t) = (-1)^{n-1}u^{(n-1)}(t)$

Lemma 4. *Let $0 < \alpha_i \leq 1$ for $1 \leq i \leq n$, $1 \leq p < \infty$, and $u \in L^p([0, T], \mathbb{R}^n)$. Then*

$$\left\| {}_0D_\xi^{-\alpha_i} u \right\|_{L^p([0,t])} \leq \frac{t^{\alpha_i}}{\Gamma(\alpha_i + 1)} \|u\|_{L^p([0,t])}, \quad \text{for } \xi \in [0, t],\ t \in [0, T]. \tag{20}$$

Proposition 5 (see [40]). *From fractional integration, we have*

$$\int_0^T [{}_0D_t^{-\alpha} u(t)] v(t) dt = \int_0^T [{}_tD_T^{-\alpha} v(t)] u(t) dt, \quad \alpha > 0, \tag{21}$$

provided that $u \in L^p([0,T],\mathbb{R})$, $v \in L^q([0,T],\mathbb{R})$, *and* $p \geq 1$, $(1/p)+(1/q) \leq 1+\alpha$ *or* $p \neq 1, q \neq 1$, $(1/p)+(1/q) = 1+\alpha$.

Definition 6 (see [40]). Let $0<\alpha_i \leq 1$ for $1 \leq i \leq n$. The fractional derivative space $H_0^{\alpha_i}(0,T)$ (denoted by $H_0^{\alpha_i}$ for short) is defined by the closure $C_0^\infty([0,T],\mathbb{R})$, that is,

$$H_0^{\alpha_i} = \overline{C_0^\infty([0,T],\mathbb{R})}, \tag{22}$$

with respect to the weighted norm

$$\|u_i\|_{\alpha_i}^p = \left(\int_0^T \left|{}_0D_t^{\alpha_i}u_i(t)\right|^p dt + \int_0^T |u_i(t)|^p\right)dt, \tag{23}$$

for every $u_i \in H_0^{\alpha_i}$ and for $1 \leq i \leq n$.

Remark 7. It is obvious that the fractional derivative space $H_0^{\alpha_i}$ is the space of functions $u_i \in L^p([0,T],\mathbb{R})$ having an α_i order Riemann–Liouville fractional derivative ${}_0D_t^{\alpha i}u_i \in L^p([0,T],\mathbb{R})$ and $u_i(0)=u_i(T)=0$ for $1 \leq i \leq n$. From [12] (Proposition 3.1), we know that for $0<\alpha_i \leq 1$, the space $H_0^{\alpha_i}$ is a reflexive and separable Banach space.

Lemma 8 (see [40]). *Let* $0<\alpha_i \leq 1$ *for* $1 \leq i \leq n$, *and* $1<p<\infty$. *For any* $u_i \in H_0^{\alpha_i}$, *we have*

$$\|u_i\|_{L^p} \leq \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)}\left\|{}_0D_t^{\alpha_i}u_i(t)\right\|_{L^p}. \tag{24}$$

Moreover, if $\alpha_i > 1/p$, *then*

$$\|u_i\|_\infty \leq \frac{T^{\alpha_i-(1/p)}}{\Gamma(\alpha_i)((\alpha_i-1)q+1)^{1/q}}\left\|{}_0D_t^{\alpha_i}u_i(t)\right\|_{L^p}, \tag{25}$$

where $(1/p)+(1/q)=1$. *Upon using (23), we observe that*

$$\|u_i\|_{\alpha_i} = \left\|{}_0D_t^{\alpha_i}u_i(t)\right\|_{L^p} = \left(\int_0^T \left|{}_0D_t^{\alpha_i}u_i(t)\right|^p dt\right)^{1/p}, \quad \forall u_i \in H_0^{\alpha_i}, \tag{26}$$

for $1 \leq i \leq n$, *which is equivalent to (15). Then, we have*

$$\sum_{i=1}^n \|u_i\|_{L^p}^p \leq S\sum_{i=1}^n \|u_i\|_{\alpha_i}^p, \tag{27}$$

and if $\alpha_i > 1/p$, *then*

$$\sum_{i=1}^n \|u_i\|_\infty^p \leq M\sum_{i=1}^n \|u_i\|_{\alpha_i}^p, \tag{28}$$

with

$$S = \max\left\{\frac{T^{p\alpha_i}}{(\Gamma(\alpha_i+1))^p}, 1 \leq i \leq n\right\},$$
$$M = \max\left\{\frac{T^{p\alpha_i-1}}{(\Gamma(\alpha_i+1))^p((\alpha_i-1)q+1)^{p/q}}, 1 \leq i \leq n\right\}. \tag{29}$$

Now, we let X be the Cartesian product of n Sobolev spaces $H_0^{\alpha_i}, \cdots, H_0^{\alpha_i}$, *i.e.,* $X = H_0^{\alpha_i} \times \cdots \times H_0^{\alpha_n}$, *which is a reflexive Banach space endowed with the norm*

$$\|(u_1,\cdots,u_n)\| = \sum_{i=1}^n \|u_i\|_{\alpha_i}. \tag{30}$$

Obviously, X is compactly embedded in $(C^0([0,T]))^n$.

Definition 9. We mean by a (weak) solution of the system (1) any function $u=(u_1,\cdots,u_n) \in X$ such that

$$\begin{aligned}&\sum_{i=1}^n \int_0^T \left(\left|{}_0D_t^{\alpha_i}u_i(t)\right|^{p-2} {}_0D_t^{\alpha_i}u_i(t)\cdot {}_0D_t^{\alpha_i}v_i(t)dt\right)\\ &\quad - \sum_{i=1}^n \int_0^T h_i(u_i(t))v_i(t)dt + \sum_{i=1}^n\sum_{j=1}^m a_i(t_j)I_{ij}(u_i(t_j))v_i(t_j)\\ &\quad - \lambda\sum_{i=1}^n \int_0^T F_{u_i}(t,u)v_i(t)dt - \mu\sum_{i=1}^n \int_0^T G_{u_i}(t,u)v_i(t)dt = 0,\end{aligned} \tag{31}$$

for every $v=(v_1,\cdots,v_n) \in X$.

Put

$$H_i(x) = \int_0^x h_i(\xi)d\xi, \quad \text{for all } x \in \mathbb{R}, \tag{32}$$

for $1 \leq i \leq n$.

We need the following conditions:

(H1): $1/p < \alpha_i \leq 1$ for $1 \leq i \leq n$.

(H2): $I_{ij}(0)=0$, and there exists a constant $L_{ij}>0$ such that

$$\begin{aligned}&\left|I_{ij}(s_1)-I_{ij}(s_2)\right| \leq L_{ij}|s_1-s_2|^{p-1},\\ &\qquad \text{for any } s_1,s_2 \in \mathbb{R}\ (i=1,\cdots,n, j=1,\cdots,m).\end{aligned} \tag{33}$$

(H3): $\sum_{i=1}^n (L_iT^{p\alpha_i}/(\Gamma(\alpha_i+1))^{p\overline{a_i}}) + MCm\|\bar{a}\|_\infty < 1$, where $C=\max_{i\in\{1,\cdots,n\},ji\in\{1,\cdots,m\}}L_{ij}$ and $\bar{a}=\max\{a_i(t), t \in [0, T], 1 \leq i \leq n\}$.

Put

$$\begin{aligned}
\sigma &= \min\left\{1 - \frac{L_i T^{p\alpha_i}}{(\Gamma(\alpha_i+1))^p}, 1 \le i \le n\right\},\\
\rho &= \max\left\{1 + \frac{L_i T^{p\alpha_i}}{(\Gamma(\alpha_i+1))^{p\overline{a_i}}}, 1 \le i \le n\right\},\\
\varrho_1 &= \sigma - MCm\|\bar{a}\|_\infty,\\
\varrho_2 &= \sigma + MCm\|\bar{a}\|_\infty.
\end{aligned} \tag{34}$$

3. Main Results

In this section, we present our key findings regarding the existence of at least three weak system solutions (1). For any $\varsigma > 0$, we denote by $Q(\varsigma)$ the set $\{(x_i, \cdots, x_n) \in \mathbb{R}^n : (1/p) \sum_{i=1}^n |x_i|^p \le \varsigma\}$. For positive constants θ and η, set

$$\begin{aligned}
G^\theta &:= \int_0^T \max_{(x_1,\cdots,x_n)\in Q(\theta)} G(t, x_1, \cdots, x_n)dt,\\
G_\eta &:= \inf_{[0,T]\times[0,\Gamma(2-\alpha_1)\eta]\times\cdots\times[0,\Gamma(2-\alpha_n)\eta]} G(t, x_1, \cdots, x_n)dt.
\end{aligned} \tag{35}$$

For the rest of this article, positive constants will be used (θ and η), and let Θ and η be the vectors in $\mathbb{R}^n$ defined by

$$\begin{aligned}
\Theta &= \left(\sqrt[p]{\theta}, \cdots, \sqrt[p]{\theta}\right),\\
\bar{\eta} &= (\Gamma(2-\alpha_1)\eta, \cdots, \Gamma(2-\alpha_n)\eta),
\end{aligned} \tag{36}$$

respectively.

Set

$$\begin{aligned}
C_i(\alpha_i, \gamma) = \frac{1}{p(\gamma T)^p}\Bigg\{&\int_0^{\gamma T} t^{p(1-\alpha_i)}dt\\
&+ \int_{\gamma T}^{(1-\gamma)T} \left(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i}\right)^p dt\\
&+ \int_{(1-\gamma)T}^{T} \Big[\left(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i}\right)\\
&- 1 - ((1-\gamma)T)^{1-\alpha_i}\Big]^p\Bigg\},
\end{aligned} \tag{37}$$

for $0 < \gamma < 1/p$, and

$$\begin{aligned}
K_1 &= \max\{C_i(\alpha_i, \gamma), 1 \le i \le n\},\\
K_2 &= \min\{C_i(\alpha_i, \gamma), 1 \le i \le n\}.
\end{aligned} \tag{38}$$

Fixing four positive constants θ_1, θ_2, θ_3, and η, put

$$\begin{aligned}
\delta_{\lambda,G} := \min\Bigg\{&\frac{1}{pM}\min\Bigg\{\frac{\varrho_1\theta_1^p - pM\lambda\int_0^T F(t,\Theta_1)dt}{G^{\theta_1}},\\
&\cdot \frac{\varrho_1\theta_2^p - pM\lambda\int_0^T F(t,\Theta_2)dt}{G^{\theta_2}}, \frac{\varrho_1\left(\theta_3^p - \theta_2^p\right) - pM\lambda\int_0^T F(t,\Theta_3)dt}{G^{\theta_3}}\Bigg\}\\
&\cdot \frac{K_1 n\varrho_2\eta^p - \lambda\left(\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{n})dt - \int_0^T F(t,\Theta_1)dt\right)}{TG_\eta - G^{\theta_1}}\Bigg\},
\end{aligned} \tag{39}$$

for $0 < \gamma < 1/p$.

Theorem 10. *Let $F : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be nonnegative. Assume that there exist positive constants $\gamma < 1/p$, θ_1, θ_2, θ_3, and η with $\theta_1 < (pMK_2 n)^{1/p}\eta$ and $(pMK_1 n\varrho_2/\varrho_1)^{1/p}\eta < \theta_2 < \theta_3$ such that*

(A1):

$$\begin{aligned}
&\max\left\{\frac{\int_0^T F(t,\Theta_1)dt}{\theta_1^p}, \frac{\int_0^T F(t,\Theta_2)dt}{\theta_2^p}, \frac{\int_0^T F(t,\Theta_3)dt}{\theta_3^p - \theta_2^p}\right\}\\
&\quad < \frac{\varrho_1}{pM}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt}{K_1 n\varrho_2\eta^p}.
\end{aligned} \tag{40}$$

Then, for every

$$\begin{aligned}
\lambda \in \Bigg]&\frac{K_1 n\varrho_2\eta^p}{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt}, \frac{\varrho_1}{pM}\min\\
&\cdot\left\{\frac{\theta_1^p}{\int_0^T F(t,\Theta_1)dt}, \frac{\theta_2^p}{\int_0^T F(t,\Theta_2)dt}, \frac{\theta_3^p - \theta_2^p}{\int_0^T F(t,\Theta_3)dt},\right\}\Bigg[,
\end{aligned} \tag{41}$$

and every nonnegative function $G : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying $G^\theta \ge 0$, there exists $\delta_{\lambda,G} > 0$ given by (39) such that, for each $\mu \in [0, \delta_{\lambda,G}[$, the system (1) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]} |u_1(t)| < \theta_1$, $\max_{t\in[0,T]} |u_2(t)| < \theta_2$, and $\max_{t\in[0,T]} |u_3(t)| < \theta_3$.

Proof. Our aim is to apply Theorem 1 to the system (1). We take $X = H_0^{\alpha_1} \times \cdots \times H_0^{\alpha_n}$ and introduce the functionals Φ and Ψ for $u = (u_1, u_2, \cdots, u_3) \in X$, as follows:

$$\begin{aligned}
\Phi(u) = \frac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p - \sum_{i=1}^n \int_0^T H_i(u_i(t))dt\\
+ \sum_{i=1}^n \sum_{j=1}^m a_i(t_j)\int_0^{u_i(t_j)} I_{ij}(s)ds,
\end{aligned} \tag{42}$$

$$\Psi(u) = \int_0^T F(t, u(t))dt + \frac{\mu}{\lambda}\int_0^T G(t, u(t))dt, \tag{43}$$

and we put

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u). \tag{44}$$

Clearly, Φ and Ψ are continuously Gateaux differentiable functionals whose Gateaux derivatives at the point $u \in X$ are given by

$$\begin{aligned} \Psi'(u)(v) &= \int_0^T \sum_{i=1}^n F_{u_i}(t, u(t)) v_i(t)dt \\ &\quad + \frac{\mu}{\lambda}\int_0^T \sum_{i=1}^n G_{u_i}(t, i(t)) v_i(t)dt, \\ \Phi'(u)(v) &= \sum_{i=1}^n \left(\int_0^T a_i(t) \left|{}_0D_t^{\alpha_i} u_i(t)\right|^{p-2} {}_0D_t^{\alpha_i} u_i(t) \cdot D_t^{\alpha_i} v_i(t)dt \right) \\ &\quad - \int_0^T \sum_{i=1}^n h_i(u_i(t)) v_i(t)dt \\ &\quad + \sum_{j=1}^m \sum_{i=1}^n a_i(t_j) I_{ij}(u_i(t_j)) v_i(t_j), \end{aligned} \tag{45}$$

for every $v = (v_1, v_2, \cdots, v_n) \in X$. Clearly, $\Phi', \Psi' \in X^*$, and we easily observe that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.

We can show by (42) that Φ is sequentially weakly lower semicontinuous. Indeed, taking the sequentially weakly lower semicontinuity property of the norm into account and since H_i is continuous for $i = 1, \cdots, n$, it is enough to prove that

$$\sum_{j=1}^m \sum_{i=1}^n a_i(t_j) \int_0^{u_i(t_j)} I_{ij}(s)ds, \tag{46}$$

is weakly continuous in X. In fact, for $\{u_k = (u_{1k}, \cdots, u_{nk})\} \subset X$, if $\{u_k\}$ converges to u in X, then there exists $S_1 > 0$ such that $\|u_k\|_\infty \leq S_1$. Therefore, we have

$$\begin{aligned} &\left| \sum_{j=1}^m \sum_{i=1}^n a_i(t_j) \int_0^{u_{ik}(t_j)} I_{ij}(s)ds - \sum_{j=1}^m \sum_{i=1}^n a_i(t_j) \int_0^{u_i(t_j)} I_{ij}(s)ds \right| \\ &\quad \leq \sum_{j=1}^m \sum_{i=1}^n \left| a_i(t_j) \int_{u_i(t_j)}^{u_{ik}(t_j)} I_{ij}(s)ds \right| \\ &\quad \leq S_2 mn \|\bar{a}\|_\infty \|u_k - u\|_\infty \longrightarrow 0, \end{aligned} \tag{47}$$

where $S_2 = \max_{i\in\{1,\cdots,n\}, j\in\{,\cdots,m\}, |s|\leq S_1} I_{ij}(s)$. So, we have $|\Phi(u_k) - \Phi(u)| \longrightarrow 0$; thus, Φ is weakly continuous. Hence, Φ is sequentially weakly lower semicontinuous in X. We show what is required. Since $h_i(0) = 0$, one has $|h_i(x_i)| \leq L_i |x_i|^{p-1}$ for $i = 1, \cdots, n$; from (43) and the condition (H2), we see that

$$\begin{aligned} \frac{\varrho_1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p &\leq \frac{\sigma}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p - \frac{M}{p}\sum_{i=1}^n \sum_{j=1}^m L_{ij} \|a_i\|_\infty \|u_i\|_{\alpha_i}^p \\ &\leq \frac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p - \sum_{i=1}^n \frac{L_i T^{p\alpha_i}}{p(\Gamma(\alpha_i+1))^{p_{\overline{\alpha_i}}}} \|u_i\|_{\alpha_i}^p \\ &\quad + \sum_{i=1}^n \sum_{j=1}^m a_j(t_j) \int_0^{u_i(t_j)} I_{ij}(s)ds \leq \Phi(u) \\ &\leq \frac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p + \sum_{i=1}^n \frac{L_i T^{p\alpha_i}}{p(\Gamma(\alpha_i+1))^{p_{\overline{\alpha_i}}}} \|u_i\|_{\alpha_i}^p \\ &\quad + \sum_{i=1}^n \sum_{j=1}^m a_i(t_j) \int_0^{u_i(t_j)} I_{ij}(s)ds \\ &\leq \frac{\rho}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p + \frac{M}{p}\sum_{i=1}^n \sum_{j=1}^m L_{ij} \|a_i\|_\infty \|u_i\|_{\alpha_i}^p \\ &\leq \frac{\varrho_2}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i}^p, \end{aligned} \tag{48}$$

and bearing the condition (H3) in mind, it follows $\lim_{\|u\|\longrightarrow\infty} \Phi(u) = +\infty$; namely, Φ is coercive and convex.

For $0 < \gamma < 1/p$, define $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ by

$$\omega_i(t) = \begin{cases} \dfrac{\Gamma(2-\alpha_i)\eta}{\gamma T} t, & t \in [0, \gamma T[, \\ \Gamma(2-\alpha_i)\eta, & t \in [\gamma T, (1-\gamma)T], \\ \dfrac{\Gamma(2-\alpha_i)\eta}{\gamma T}(T-1), & t \in](1-\gamma)T, T], \end{cases} \tag{49}$$

for $1 \leq i \leq n$. Clearly, $\omega_i(0) = \omega_i(T) = 0$ and $\omega_i \in L^p([0, T])$ for $1 \leq i \leq n$. A direct calculation shows that

$${}_0D_t^{\alpha_i}\omega_i(t) = \begin{cases} \dfrac{\eta}{\gamma T} t^{1-\alpha_i}, & t \in [0, \gamma T[, \\ \dfrac{\eta}{\gamma T}\left(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i}\right), & t \in [\gamma T, (1-\gamma)T], \\ \dfrac{\eta}{\gamma T}\left(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i} - (t-(1-\gamma)T)^{1-\alpha_i}\right), & t \in](1-\gamma)T, T], \end{cases} \tag{50}$$

for $1 \le i \le n$. Furthermore,

$$\begin{aligned}\int_0^T \left|{}_0D_t^{\alpha_i}\omega_i(t)\right|^p dt &= \left(\frac{\eta}{\gamma T}\right)^p \left\{\int_0^{\gamma T} t^{(1-a_i)p} dt \right.\\ &\quad + \int_{hT}^{(1-\gamma)T} \left(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i}\right)^p dt \\ &\quad + \int_{(1-h)T}^{T} \left(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i}\right. \\ &\quad \left.\left. - (t-(1-\gamma)T)^{1-\alpha_i}\right)^p dt\right\} \\ &= p\eta^p C_i(\alpha_i, \gamma),\end{aligned} \tag{51}$$

for $1 \le i \le n$. Thus, $\omega \in X$, and

$$\|\omega_i\|_{\alpha_i}^p = p\eta^p C(\alpha_i, \gamma), \tag{52}$$

for $1 \le i \le n$. By using (50) and (52), we have

$$K_2 n \varrho_1 \eta^p \le \Phi(\omega) \le K_1 n \varrho_1 \eta^p. \tag{53}$$

Choose $r_1 = (\varrho_1/pM)\theta_1^p$, $r_2 = (\varrho_1/pM)\theta_2^p$, and $r_3 = (\varrho_1/pM)(\theta_3^p - \theta_2^p)$. From the conditions $\theta_3 > \theta_2$, $\theta_1 < (pMK_2 n)^{1/p}\eta$, and $(pMK_1 n\varrho_2/\varrho_1)^{1/p}\eta < \theta_2$, we achieve $r_3 > 0$ and $r_1 < \Phi(\omega) < r_2$. From the definition of Φ and considering Equations (24), (27), and (50), one has

$$\begin{aligned}\Phi^{-1}(-\infty; r_1) &= \{u \in X : \Phi(u) \le r_1\} \\ &\subseteq \left\{u \in X : \sum_{i=1}^n \|u_i\|_{\alpha_i}^p \le \frac{pr_1}{\varrho_1}\right\} \\ &= \left\{u \in X : \frac{1}{M}\sum_{i=1}^n \|u_i\|_{\infty}^p \le \frac{pr_1}{\varrho_1}\right\} \\ &= \left\{u \in X : \sum_{i=1}^n \|u_i\|_{\alpha_i}^p \le \frac{pMr_1}{\varrho_1}\right\} \\ &= \left\{u \in X : \sum_{i=1}^n \|u_i\|_{\infty}^p \le \theta_1^p\right\}.\end{aligned} \tag{54}$$

Hence, since F is nonnegative, one has

$$\begin{aligned}&\sup_{u \in \Phi^{-1}(-\infty; r_1)} \int_0^T F(t, u(t)) \\ &\le \int_0^T \max_{(x_1, x_2, \cdots, x_n) \in Q(\theta_1)} F(t, x_1, x_2, \cdots, x_n) dt \\ &\le \int_0^T F(t, \Theta_1) dt.\end{aligned} \tag{55}$$

In a similar way, we have

$$\begin{aligned}&\sup_{u \in \Phi^{-1}(-\infty; r_2)} \int_0^T F(t, u(t)) \le \int_0^T F(t, \Theta_2) dt, \\ &\sup_{u \in \Phi^{-1}(-\infty; r_2 + r_3)} \int_0^T F(t, u(t)) \le \int_0^T F(t, \Theta_3) dt.\end{aligned} \tag{56}$$

Therefore, since $0 \in \Phi^{-1}(-\infty; r_1)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}\varphi(r_1) &:= \inf_{u \in \Phi^{-1}(-\infty, r_1)} \frac{\left(\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)\right) - \Psi(u)}{r_1 - \Phi(u)} \le \frac{\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)}{r_1} \\ &= \frac{\sup_{u \in \Phi^{-1}(-\infty, r_1)} \int_0^T [F(t, u(t)) + (\mu/\lambda) G(t, u(t))] dt}{r_1} \le \frac{pM}{\varrho_1} \cdot \frac{\int_0^T F(t, \Theta_1) dt + (\mu/\lambda) G^{\theta_1}}{\theta_1^p},\end{aligned} \tag{57}$$

$$\left\{\varphi(r_2) \le \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2)} \Psi(u)}{r_2} = \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2)} \int_0^T [F(t, u(t)) + (\mu/\lambda) G(t, u(t))] dt}{r_2} \le \frac{pM}{\varrho_1} \cdot \frac{\int_0^T F(t, \Theta_2) dt + (\mu/\lambda) G^{\theta_2}}{\theta_2^p},\right. \tag{58}$$

$$\left\{\gamma(r_2, r_3) = \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2 + r_3)} \Psi(u)}{r_3} = \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2 + r_3)} \int_0^T [F(t, u(t)) + (\lambda/\mu) G(t, u(t))] dt}{r_3} \le \frac{pM}{\varrho_1} \cdot \frac{\int_0^T F(t, \Theta_3) dt + (\mu/\lambda) G^{\theta_3}}{\theta_3^p - \theta_2^p}.\right. \tag{59}$$

On the other hand, for each $u \in \Phi - 1(-\infty, r_1)$, one has

$$\left\{\beta(r_1, r_2) \geq \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt + (\mu/\lambda)\left(TG^{\eta} - G^{\theta_1}\right)}{\Phi(\omega) - \Phi(u)} \geq \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt + (\mu/\lambda)\left(TG_{\eta} - G^{\theta_1}\right)}{K_1 n \varrho_2 \eta^p}. \right. \tag{60}$$

Since $\mu < \delta_{\lambda,G}$, one has

$$\mu < \frac{1}{pM} \frac{\varrho_1 \theta_1^p - pM\lambda \int_0^T F(t,\Theta_1)dt}{G^{\theta_1}}. \tag{61}$$

This means

$$\frac{\int_0^T F(t,\Theta_1)dt + (\mu/\lambda)G^{\theta_1}}{(\varrho_1/pM)\theta_1} < \frac{1}{\lambda}. \tag{62}$$

Furthermore,

$$\mu < \frac{K_1 n \varrho_2 \eta^p - \lambda\left(\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt\right)}{TG_{\eta} - G^{\theta_1}}. \tag{63}$$

This means

$$\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt + (\mu/\lambda)\left(TG_{\eta} - G^{\theta_1}\right)}{K_1 n \varrho_2 \eta^p} > \frac{1}{\lambda}. \tag{64}$$

Then,

$$\frac{\int_0^T F(t,\Theta_1)dt + (\mu/\lambda)G^{\theta_1}}{(\varrho_1/pM)\theta_1^p} < \frac{1}{\lambda} < \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt + (\mu/\lambda)\left(TG_{\eta} - G^{\theta_1}\right)}{K_1 n \varrho_2 \eta^p}. \tag{65}$$

In a similar way, we have

$$\frac{\int_0^T F(t,\Theta_2)dt + (\mu/\lambda)G^{\theta_2}}{(\varrho_1/pM)\theta_2^p} < \frac{1}{\lambda} < \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt + (\mu/\lambda)\left(TG_{\eta} - G^{\theta_1}\right)}{K_1 n \varrho_2 \eta^p}, \tag{66}$$

$$\frac{\int_0^T F(t,\Theta_3)dt + (\mu/\lambda)G^{\theta_3}}{(\varrho_1/pM)\left(\theta_3^p - \theta_2^p\right)} < \frac{1}{\lambda} < \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt + (\mu/\lambda)\left(TG_{\eta} - G^{\theta_1}\right)}{K_1 n \varrho_2 \eta^p}. \tag{67}$$

Hence, from (57)–(67), we get

$$\alpha(r_1, r_2, r_3) < \beta(r_1, r_2). \tag{68}$$

Therefore, (a_1) and (a_2) of Theorem 1 are verified.

Now, we show that the functional I_λ satisfies the assumption (a_2) of Theorem 1. Let $u^* = (u_1^*, u_2^*, \cdots, u_n^*)$ and $u^{**} = (u_1^{**}, u_2^{**}, \cdots, u_n^{**})$ be two local minima for I_λ. Then, u^* and u^{**} are critical points for I_λ; they are weak solutions for the system (1). Since we assumed F is nonnegative and since G is nonnegative, for fixed $\lambda > 0$ and $\mu \geq 0$, we have $F(t, su^* + (1-s)u^{**}) + (\mu/\lambda)G(t, su^* + (1-s)u^{**}) \geq 0$, and consequently, $\Psi(t, su^* + (1-s)u^{**}) \geq 0$ for all $s \in [0,1]$. Hence, Theorem 1 implies that for every

$$\lambda \in \left] \frac{K_1 n \varrho_2 \eta^p}{\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt}, \frac{\varrho_1}{pM} \min \cdot \left\{ \frac{\theta_1^p}{\int_0^T F(t,\Theta_1)dt}, \frac{\theta_2^p}{\int_0^T F(t,\Theta_2)dt}, \frac{\theta_3^p - \theta_2^p}{\int_0^T F(t,\Theta_3)dt} \right\} \right[, \tag{69}$$

and $\mu \in [0, \delta_\lambda, G[$, the functional I_λ has three critical points u_i, $i = 1, 2, 3$, in X such that $\Phi(u_1) < r_1$, $\Phi(u_2) < r_2$, and $\Phi(u_3) < r_2 + r_3$, that is, $\max_{t\in[0,T]}|u_1(t)| < \theta_1$, $\max_{t\in[0,T]}|u_2(t)| < \theta_2$, and $\max_{t\in[0,T]}|u_3(t)| < \theta_3$. Then, taking into account the fact that the weak solutions of the system (1) are exactly critical points of the functional I_λ, we have the desired conclusion. □

For positive constants θ_1, θ_4, and η, set

$$\begin{aligned} \delta'_{\lambda,G} := \min \Bigg\{ & \frac{1}{pM} \min \Bigg\{ \frac{\varrho_1 \theta_1^p - pM\lambda \int_0^T F(t,\Theta_1)dt}{G^{\theta_1}}, \\ & \cdot \frac{\varrho_1 \theta_4^p - 2pM\lambda \int_0^T F\left(\left(t, \theta_4/\sqrt[p]{2}, \theta_4/\sqrt[p]{2}, \cdots, \theta_4/\sqrt[p]{2}\right)\right)dt}{G\left(\theta_4/\sqrt[p]{2}\right)}, \\ & \cdot \frac{\varrho_1 \theta_4 - 2pM\lambda \int_0^T F(t,\Theta_4)dt}{G^{\theta_4}} \Bigg\}, \\ & \cdot \frac{K_1 n \varrho_2 \eta^p - \lambda\left(\int_{\gamma T}^{(1-\gamma)T} F(t,\bar{\eta})dt - \int_0^T F(t,\Theta_1)dt\right)}{TG_{\eta} - G^{\theta_1}} \Bigg\}, \end{aligned} \tag{70}$$

where $0 < \gamma < 1/p$.

Theorem 11. *Let $F : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfy the condition $F(t, x_1, x_2, \cdots, x_n) \geq 0$ for all $(t, x_1, x_2, \cdots, x_n) \in [0, T] \times \mathbb{R}^n$. Assume that there exist positive constants $\gamma < 1/p$, θ_1, θ_4, and η with $\theta_1 < \min\{\eta, (pMK_2 n)^{1/p}\eta\}$ and $(2pMK_1 n\varrho_2/(\sigma - MCm\|\tilde{a}\|_\infty))^{1/p}\eta < \theta_4$ such that*

(A2):

$$\max\left\{\frac{\int_0^T F(t,\Theta_1)dt}{\theta_1^p}, \frac{2\int_0^T F(t,\Theta_4)dt}{\theta_4^p}\right\} < \frac{\varrho_1}{\varrho_1 + pMK_1 n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})dt}{\eta^p}. \tag{71}$$

Then, for every

$$\lambda \in \Lambda' := \left]\frac{(\varrho_1 + pMK_1 n\varrho_2)\eta^p}{pM\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})dt}, \frac{\varrho_1}{pM}\min\cdot\left\{\frac{\theta_1^p}{\int_0^T F(t,\Theta_1)dt}, \frac{\theta_4^p}{2\int_0^T F(t,\Theta_4)dt}\right\}\right[, \tag{72}$$

and every nonnegative function $G : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying $G^\theta \geq 0$, there exists $\delta'_{\lambda,G}$ given by (70) such that, for each $\mu \in [0, \delta_{\lambda,G}[$, the system (1) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]} |u_1(t)| < \theta_1$, $\max_{t\in[0,T]} |u_2(t)| < \theta_4/\sqrt[p]{2}$, and $\max_{t\in[0,T]} |u_3(t)| < \theta_4$. □

Proof. Choose $\theta_2 = \theta_4/\sqrt[p]{2}$ and $\theta_3 = \theta_4$. So, by using (A2), one has

$$\left\{\frac{\int_0^T F(t,\Theta_2)dt}{\theta_2^p} = \frac{2\int_0^T F\left(t, \theta_4/\sqrt[p]{2}, \theta_4/\sqrt[p]{2}, \cdots, \theta_4/\sqrt[p]{2}\right)dt}{\theta_2^p} \leq \frac{2\int_0^T (t,\Theta_4)dt}{\theta_4^p} < \frac{\varrho_1}{\varrho_1 + pMK_1 n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)} F(t,\overline{\eta})dt}{\eta^p},\right. \tag{73}$$

$$\left\{\frac{\int_0^T F(t,\Theta_3)dt}{\theta_3^p - \theta_2^p} = \frac{2\int_0^T F(t,\Theta_4)dt}{\theta_4^p} < \frac{\varrho_1}{\varrho_1 + pMK_1 n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})dt}{\eta^p}.\right. \tag{74}$$

Moreover, taking into account that $\theta_1 < \eta$, by using (A2), we have

$$\begin{aligned}
&\frac{\varrho_1}{pMK_1 n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})dt - \int_0^T F(t,\Theta_1)dt}{\eta^p} \\
&\quad > \frac{\varrho_1}{pMK_1 n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})dt}{\eta^p} - \frac{\varrho_1}{pMK_1 n\varrho_2}\frac{\int_0^T F(t,\Theta_1)dt}{\theta_1^p} \\
&\quad > \frac{\varrho_1}{pMK_1 n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})dt}{\eta^p} \\
&\qquad - \frac{\varrho_1^2}{pMK_1 n\varrho_2(\varrho_1 + pMK_1 n\varrho_2)}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})dt}{\eta^p} \\
&\quad = \frac{\varrho_1}{\varrho_1 + pMK_1 n\varrho_2}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\overline{\eta})dt}{\eta^p}.
\end{aligned} \tag{75}$$

Hence, from (A2), (73), and (74), it is easy to see that the assumption (A1) of Theorem 10 is satisfied, and since the critical points of the functional $\Phi - \lambda\Psi$ are the weak solutions of the system (1), we have the conclusion. □

Now, we present the following example in which the hypotheses of Theorem 11 are satisfied.

Example 1. Consider the following system:

$$\begin{cases}
{}_tD_1^{0,65}\left(\phi_3\left({}_0D_t^{0,65}u_1(t)\right)\right) = \lambda F_{u_1}(u_1, u_2) + \mu G_{u_1}(u_1, u_2) + h_1(u_1), & t \in [0, 1],\ t \neq t_1, \\
{}_tD_1^{0,7}\left(\phi_3\left({}_0D_t^{0,7}u_2(t)\right)\right) = \lambda F_{u_2}(u_1, u_2) + \mu G_{u_2}(u_1, u_2) + h_2(u_2), & t \in [0, 1],\ t \neq t_2, \\
\Delta\left({}_tD_T^{-0,35}\phi_3\left({}_0^cD_t^{0,65}u_1(t)\right)(t_j)\right) = I_{1j}\left(u_1(t_j)\right), & j = 1, 2, \\
\Delta\left({}_tD_T^{-0,3}\phi_3\left({}_0^cD_t^{0,7}u_2(t)\right)(t_j)\right) = I_{2j}\left(u_2(t_j)\right), & j = 1, 2, \\
u_1(0) = u_1(1) = 0, \\
u_2(0) = u_2(1) = 0,
\end{cases} \tag{76}$$

where

$$F(x_1, x_2) = \begin{cases}
e^{-1/|x_1|} + e^{-1/|x_2|} & \text{if } x_1 x_2 \neq 0, \\
e^{-1/|x_2|} & \text{if} x_1 = 0,\ x_2 \neq 0, \\
e^{-1/|x_1|} & \text{if} x_1 \neq 0,\ x_2 = 0, \\
0 & \text{if} x_1 = 0,\ x_2 = 0.
\end{cases} \tag{77}$$

$h_1(x_1) = (1/10^2)\sin^2 x_1$ and $h_2(x_2) = (1/10^3)(1 - \cos^2 x_2)$ for every $(x_1, x_2) \in \mathbb{R}$, $t_1 = 1/3$, $t_2 = 1/2$, and $I_{ij}(\xi) = (1/10^2)\xi^2$ for every $\xi \in \mathbb{R}$ and for $i, j = 1, 2$. By expressions of h_1 and h_2, we have $H_1(x_1) = (1/2.10^2)(x_1 - (1/2)\sin 2x_1)$ and $H_2(x_2) = (1/2.10^3)(2x_1 + (1/2)\sin 2x_1)$ for every $(x_1, x_2) \in \mathbb{R}$. Choosing $\gamma = 1/4$, $\theta_1 = 10^{-4}$, $\theta_4 = 10^8$, and $\eta = 1$, we clearly observe that all assumptions of Theorem 11 are satisfied. Hence, for every

$$\lambda \in \left] \frac{0.5(\Gamma(0.65))^2\left(1-\left(1/\left(10^6(\Gamma(1.65))^2\right)\right)-\left(1/\left(5\times 10^5(\Gamma(0.65))^2\right)\right)\right)}{e^{-1/\Gamma(1.35)}+e^{-1/\Gamma(1.3)}} + \frac{31.8304\left(1+\left(1/\left(10^6(\Gamma(1.65))^2\right)\right)+\left(1/\left(5\times 10^5(\Gamma(0.65))^2\right)\right)\right)}{e^{-1/\Gamma(1.35)}+e^{-1/\Gamma(1.3)}}, \cdot \frac{1-\left(1/\left(10^6(\Gamma(1.65))^2\right)\right)-\left(1/\left(5\times 10^5(\Gamma(0.65))^2\right)\right)}{(\Gamma(0.65))^2}\frac{10^{24}}{4e^{-1/10^4}} \right[, \tag{78}$$

and every nonnegative function $G:\mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying $G^\theta \geq 0$, there exists $\delta'_{\lambda,G}>0$ such that, for each $\mu \in [0,\delta_{\lambda,G}[$, the system (74) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]}|u_1(t)|<10^{-4}$, $\max_{t\in[0,T]}|u_2(t)|<10^8/\sqrt[3]{2}$, and $\max_{t\in[0,T]}|u_3(t)|<10^8$.□

Remark 12. When F does not depend on t, in Theorem 10, the assumption (A1) can be written as

$$\max\left\{\frac{F(\Theta_1)}{\theta_1^p},\frac{F(\Theta_2)}{\theta_2^p},\frac{F(\Theta_3)}{\theta_3^p-\theta_2^p}\right\} < \frac{\varrho_1}{pM}\frac{(1-2\gamma)TF(\bar{\eta})-TF(\Theta_1)}{K_1 n\varrho_2\eta^p}, \tag{79}$$

as well as

$$\Lambda := \left] \frac{K_1 n\varrho_2\eta^p}{(1-2\gamma)TF(\bar{\eta})-TF(\Theta_1)}, \frac{\varrho_1}{pTM}\min \cdot \left\{\frac{\theta_1^p}{F(\Theta_1)},\frac{\theta_2^p}{F(\Theta_2)},\frac{\theta_3^p-\theta_2^p}{F(\Theta_3)}\right\}\right[,$$

$$\delta_{\lambda,G} := \min\left\{\frac{1}{pM}\min\left\{\frac{\varrho_1\theta_1^p-pM\lambda F(\Theta_1)}{G^{\theta_1}}, \cdot\frac{\varrho_1\theta_2^p-pM\lambda F(\Theta_2)}{G^{\theta_2}},\frac{\varrho_1\left(\theta_3^p-\theta_2^p\right)-pM\lambda F(\Theta_3)}{G^{\theta_3}}\right\}, \cdot\frac{K_1 n\varrho_2\eta^p-\lambda((1-2\gamma)TF(\bar{\eta})-TF(\Theta_1))}{TG_\eta-G^{\theta_1}}\right\}. \tag{80}$$

In this case, in Theorem 11, the assumption (A2) follows the form

$$\max\left\{\frac{F(\Theta_1)}{\theta_1^p},\frac{2F(\Theta_4)}{\theta_2^p}\right\} < \frac{\varrho_1}{\varrho_1+pMK_1n\varrho_2}\frac{(1-2\gamma)TF(\bar{\eta})}{\eta^p}, \tag{81}$$

as well as

$$\Lambda' := \left] \frac{(\varrho_1+pMK_1n\varrho_2)\eta^p}{pM(1-2\gamma)TF(\bar{\eta})}, \frac{\varrho_1}{pTM}\min\left\{\frac{\theta_1^p}{F(\Theta_1)},\frac{\theta_4^p}{2F(\Theta_4)}\right\}\right[,$$

$$\delta'_{\lambda,G} := \min\left\{\frac{1}{pM}\min\left\{\frac{\varrho_1\theta_1^p-pMT\lambda F(\Theta_1)}{G^{\theta_1}}, \cdot\frac{\varrho_1\theta_4^p-2pMT\lambda F\left(t,\theta_4/\sqrt[p]{2},\theta_4/\sqrt[p]{2},\cdots,\theta_4/\sqrt[p]{2}\right)}{G^{\theta_2}}, \cdot\frac{\varrho_1\theta_4^p-2pTM\lambda F(\Theta_4)}{G^{\theta_3}}\right\}, \cdot\frac{K_1n\varrho_2\eta^p-\lambda((1-2\gamma)TF(\bar{\eta})-TF(t,\Theta_1))}{TG_\eta-G^{\theta_1}}\right\}. \tag{82}$$

Remark 13. We observe that, in our results, no asymptotic conditions on F and G are needed and only algebraic conditions on F are imposed to guarantee the existence of solutions. Moreover, in the conclusions of the above results, one of the three solutions may be trivial since the values of $F_{x_i}(t,0,0,\cdots,0)$ and $G_{x_i}(t,0,0,\cdots,0)$ for every $t\in[0,T]$, $1\leq I\leq n$, are not determined.

As an application of Theorem 10, we consider the following problem:

$$\begin{cases} {}_tD_T^\alpha\left(\phi_p\left({}_0D_t^\alpha u(t)\right)\right)=\lambda f(t,u)+\mu g(t,u)+h(u), & t\in[0,T],\ t\neq t_j, \\ \Delta\left({}_tD_T^{\alpha-1}\phi_p\left({}_0^cD_t^\alpha u_1(t)\right)(t_j)\right)=I_j\left(u(t_j)\right), & j=1,2, \\ u(0)=u(T)=0, \end{cases} \tag{83}$$

where $1/p<\alpha<1$, $\lambda>0$, $\mu\geq 0$, $T>0$, ${}_0D_t^\alpha$ and ${}_tD_T^{\alpha_i}$ denote the left and right Riemann–Liouville fractional derivatives of order α, respectively, $a_0=\operatorname{ess\,inf}_{t\in[0,T]}a(t)$, $f,g:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ are L^1-Caratheodory functions, and $h:\mathbb{R}\longrightarrow[0,+\infty)$ is a $(p-1)$-Lipschitz continuous function with the Lipschitz constant $L>0$, i.e.,

$$|h(x_1)-h(x_2)|\leq L|x_1-x_2|^{p-1}, \tag{84}$$

for every $x_1,x_2\in\mathbb{R}$, satisfying $h(0)=0$, $I_j\in C(\mathbb{R},\mathbb{R})$ for $j=1,2,\cdots,m$, such that $I_j(0)=0$, $0<t_1<t_2<\cdots<t_m<t_{m+1}=T$, and I_j is a $(p-1)$-Lipschitz continuous function with the Lipschitz constant $L_j>0$ such that

$$\left|I_j(s_1)-I_j(s_2)\right|\leq L_j|s_1-s_2|^{p-1}, \tag{85}$$

for any $s_1, s_1 \in \mathbb{R}$, satisfying $I_j(0) = 0$, for $j = 1, 2, \cdots, m$. Put

$$F(t,x) = \int_0^x f(t,\xi)dt, \quad \text{for every}(t,x) \in [0,T]\times\mathbb{R},$$
$$G(t,x) = \int_0^x g(t,\xi)dt, \quad \text{for every}(t,x) \in [0,T]\times\mathbb{R}, \tag{86}$$

and $H(x) = \int_0^x h(\xi)dt$ for every $x \in \mathbb{R}$.

Set

$$\begin{aligned}
\bar{\sigma} &= 1 - \frac{LT^{p\alpha}}{(\Gamma(\alpha+1))^p},\\
\bar{\rho} &= 1 + \frac{LT^{p\alpha}}{(\Gamma(\alpha+1))^p \bar{a}},\\
\bar{M} &= \frac{T^{p\alpha-1}}{\alpha_0(\Gamma(\alpha))^p((\alpha-1)q+1)^{p/q}},\\
\bar{\varrho}_1 &= \sigma - MCm\|\bar{a}\|_\infty,\\
\bar{\varrho}_2 &= \sigma + MCm\|\bar{a}\|_\infty,\\
C(\alpha,\gamma) &= \frac{1}{p(\gamma T)^p}\Bigg\{\int_0^{\gamma T} t^{p(1-\alpha)}dt\\
&\quad + \int_{\gamma T}^{(1-\gamma)T}\left(t^{1-\alpha} - (t-\gamma T)^{1-\alpha}\right)^p dt\\
&\quad + \int_{(1-\gamma)T}^{T}\left[\left(t^{1-\alpha} - (t-\gamma T)^{1-\alpha}\right)\right.\\
&\quad \left. - 1 - ((1-\gamma)T)^{1-\alpha}\right]^p\Bigg\},
\end{aligned} \tag{87}$$

for $0 < \gamma < 1/p$. We suppose that

$$\bar{K} = \frac{L_i T^{p\alpha}}{(\Gamma(\alpha+1))^p \bar{\alpha}_i} + \overline{MC}m\|\bar{a}\|_\infty < 1, \tag{88}$$

where $\bar{C} = \max_{j\in\{1,2,\cdots,m\}}\{L_j\}$.

For positive constants θ and η, set

$$G^\theta := \int_0^T \max_{|x|\le\sqrt[p]{\theta}} G(t,x)dt,$$
$$G_\eta := \inf_{[0,T]\times[0,\Gamma(2-\alpha)]} G. \tag{89}$$

Obviously, if g changes sign on $[0,T]$, then clearly $G^\theta \geq 0$.□

Now, we give the following straightforward consequences of Theorems 10 and 11, respectively.

Theorem 14. *Let $f : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ be a nonnegative L^1-Caratheodory function. Assume that there exist positive constants $\gamma < 1/p$, θ_1, θ_2, θ_3, and η with $\theta_3 > \theta_2$, $\theta_1 < (p\bar{M}C(\alpha,\gamma))^{1/p}\eta$, and $(p\bar{M}C(\alpha,\gamma)\bar{\varrho}_2/\bar{\varrho}_1)^{1/p}\eta < \theta_2$ such that*

$$\begin{aligned}
&\max\left\{\frac{\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt}{\theta_1^p}, \frac{\int_0^T F\left(t,\sqrt[p]{\theta_2}\right)dt}{\theta_2^p}, \frac{\int_0^T F\left(t,\sqrt[p]{\theta_3}\right)dt}{\theta_3^p - \theta_2^p}\right\}\\
&\quad < \frac{\bar{\varrho}_1}{p\bar{M}C(\alpha,\gamma)\bar{\varrho}_2}\frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha)\eta)dt - \int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt}{\eta^p}.
\end{aligned} \tag{90}$$

Then, for every

$$\begin{aligned}
\lambda \in \Lambda'' &:= \Bigg[C(\alpha,\gamma)\bar{\varrho}_2 \frac{\eta^p}{\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha)\eta)dt - \int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt},\\
&\quad \cdot \frac{\bar{\varrho}_1}{p\bar{M}}\min\Bigg\{\frac{\theta_1^p}{\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt},\\
&\quad \cdot \frac{\theta_2^p}{\int_0^T F\left(t,\sqrt[p]{\theta_2}\right)dt}, \frac{\theta_3^p - \theta_2^p}{\int_0^T F\left(t,\sqrt[p]{\theta_3}\right)dt}\Bigg\}\Bigg],
\end{aligned} \tag{91}$$

and every nonnegative L^1-Caratheodory function $g : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$, there exists $\delta^*_{\lambda,g} > 0$ given by

$$\begin{aligned}
\delta^*_{\lambda,g} &:= \min\Bigg\{\frac{1}{p\bar{M}}\min\Bigg\{\frac{\bar{\varrho}_1\theta_1^p - p\bar{M}\lambda\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt}{G^{\theta_1}},\\
&\quad \cdot \frac{\varrho_1\theta_2^p - pM\lambda\int_0^T F\left(t,\sqrt[p]{\theta_2}\right)dt}{G^{\theta_2}}, \frac{\bar{\varrho}_1(\theta_3^p - \theta_2^p) - p\bar{M}\lambda\int_0^T F\left(t,\sqrt[p]{\theta_3}\right)dt}{G^{\theta_3}}\Bigg\},\\
&\quad \cdot \frac{C(\alpha,\gamma)\bar{\varrho}_2\eta^p - \lambda\left(\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha)\bar{\eta})dt - \int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt\right)}{TG_\eta - G^{\theta_1}}\Bigg\},
\end{aligned} \tag{92}$$

such that, for each $\mu \in [0,\delta^*_{\lambda,G}[$, the system (83) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]}|u_1(t)| < \theta_1$, $\max_{t\in[0,T]}|u_2(t)| < \theta_2$, and $\max_{t\in[0,T]}|u_3(t)| < \theta_3$.□

Proof. By a similar argument as given in the proof of Theorem 10, we ensure the existence of the weak solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]}|u_1(t)| < \theta_1$, $\max_{t\in[0,T]}|u_2(t)| < \theta_2$, and $\max_{t\in[0,T]}|u_3(t)| < \theta_3$. Now, we show that the weak solutions u_1, u_2, and u_3 are nonnegative. To this end, let u_0 be a nontrivial weak solution of the problem (83). Arguing by a contradiction, assume that the set $\mathscr{A} = \{t \in]0,T] : u_0(t) < 0\}$ is nonempty and of positive measure. Put $\bar{v}(t) = \min\{0, u_0(t)\}$ for all $t \in [0,T]$. Clearly, $\bar{v} \in H^\alpha$, and one has

$$\begin{aligned}&\int_0^T \left(a(t)|{}_0D_t^{\alpha}u_0(t)|^{p-2}{}_0D_t^{\alpha}u_0(t)\cdot{}_0D_t^{\alpha}\bar{v}(t)dt\right)\\&\quad-\int_0^T h(u_0(t))\bar{v}(t)+\sum_{j=1}^{m}a(t_j)I_{ij}(u_0(t_j))\bar{v}(t_j)\\&\quad-\lambda\int_0^T f(t,u_0)\bar{v}(t)dt-\mu\int_0^T g(t,u_0)\bar{v}(t)dt=0.\end{aligned}\tag{93}$$

Thus, from our sign assumptions on the data, we have

$$\begin{aligned}0&\le\left(1-\bar{K}\right)\int_{\mathscr{A}}a(t)|{}_0D_t^{\alpha}u_0(t)|^p dt\\&\le\int_{\mathscr{A}}a(t)|{}_0D_t^{\alpha}u_0(t)|^p dt-\int_{\mathscr{A}}h(u_0(t))u_0(t)dt\\&\quad+\sum_{j=1}^{m}a(t_j)I_{ij}(u_0(t_j))u_0(t)\le 0.\end{aligned}\tag{94}$$

Hence, since $\bar{K}<1$, $u_0=0$ in A and we arrive at a contradiction. □

Theorem 15. *Let $f:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ be a nonnegative L^1-Caratheodory function. Assume that there exist positive constants $\gamma<1/p$, θ_1, θ_4, and η with $\theta_1<\min\{\eta,(p\bar{M}C(\alpha,\gamma))^{1/p}\eta\}$ and $(2p\bar{M}C(\alpha,\gamma)\bar{\varrho}_2/\bar{\varrho}_1)^{1/p}\eta<\theta_4$ such that*

$$\begin{aligned}&\max\left\{\frac{\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt}{\theta_1^p},\frac{2\int_0^T F\left(t,\sqrt[p]{\theta_4}\right)dt}{\theta_4^p}\right\}\\&\quad<\frac{\bar{\varrho}_1}{\bar{\varrho}_1+p\bar{M}C(\alpha,\gamma)\bar{\varrho}_2}\frac{\int_{\gamma T}^{(1-\gamma)T}F(t,\Gamma(2-\alpha)\eta)dt}{\eta^p}.\end{aligned}\tag{95}$$

Then, for every

$$\begin{aligned}\lambda\in&\left]\frac{\left(\bar{\varrho}_1+p\bar{M}C(\alpha,\gamma)\bar{\varrho}_2\right)\eta^p}{p\bar{M}\int_{\gamma T}^{(1-\gamma)T}F(t,\Gamma(2-\alpha)\eta)dt},\frac{\bar{\varrho}_1}{p\bar{M}}\min\right.\\&\left.\cdot\left\{\frac{\theta_1^p}{\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt},\frac{\theta_4^p}{2\int_0^T F\left(t,\sqrt[p]{\theta_4}\right)dt}\right\}\right[,\end{aligned}\tag{96}$$

and every nonnegative function $g:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfying $G^{\theta}\ge 0$, there exists $\delta_{\lambda,g}^{**}>0$ given by

$$\begin{aligned}\delta_{\lambda,g}^{**}:=\min&\left\{\frac{1}{p\bar{M}}\left\{\frac{\bar{\varrho}_1\theta_1^p-p\bar{M}\lambda\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt}{G^{\theta_1}},\right.\right.\\&\left.\cdot\frac{\bar{\varrho}_1\theta_4^p-2p\bar{M}\lambda\int_0^T F\left(t,\theta_4/\sqrt[p]{2}\right)dt}{2G^{\theta_4/\sqrt[p]{2}}},\frac{\bar{\varrho}_1\theta_4^p-2P\bar{M}\lambda\int_0^T F(t,\theta_4)dt}{2G^{\theta_4}}\right\},\\&\left.\cdot\frac{C(\alpha,\gamma)\bar{\varrho}_2\eta^p-\lambda\left(\int_{\gamma T}^{(1-\gamma)T}F(t,\Gamma(2-\alpha)\eta)-\int_0^T F\left(t,\sqrt[p]{\theta_1}\right)dt\right)}{TG_\eta-G^{\theta_1}}\right\},\end{aligned}\tag{97}$$

such that, for each $\mu\in[0,\delta_{\lambda,g}^{**}[$, the system (83) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]}|u_1(t)|<\theta_1$, $\max_{t\in[0,T]}|u_2(t)|<\theta_4/\sqrt[p]{2}$, and $\max_{t\in[0,T]}|u_3(t)|<\theta_4$. □

Here, in order to illustrate Theorem 15, we present the following example.

Example 2.

$$\begin{cases}{}_tD_1^{0.8}\left(\phi_3\left({}_0D_t^{0.8}u(t)\right)\right)=\lambda f(u)+\mu g(u)+h(u), & t\in[0,1],\ t\neq t_j,\\ \Delta\left({}_tD_1^{-0.2}\phi_3\left({}_0^cD_t^{0.8}u_1(t)\right)(t_j)\right)=I_j\left(u(t_j)\right), & j=1,2,\\ u(0)=u(1)=0,\end{cases}\tag{98}$$

where

$$f(x)=\begin{cases}6x^5, & x\le 1,\\ \dfrac{6}{x}, & x>1.\end{cases}\tag{99}$$

$h(x)=(1/10^8)\sin^2 x$, for every $x\in\mathbb{R}$, $t_1=1/4$, $t_2=1/5$, and $I_j(\xi)=(1/10^8)\xi^2$, for every $\xi\in\mathbb{R}$ and for $j=1,2$. By expression, we have

$$F(x)=\begin{cases}x^6, & x\le 1,\\ 6\ln(x)+1, & x>1.\end{cases}\tag{100}$$

Taking $\gamma=1/4$, $\theta_1=10^{-4}$, $\theta_4=10^4$, and $\eta=1$, we clearly observe that all assumptions of Theorem 15 are satisfied.

Then, for each

$$\begin{aligned}\lambda\in&\left]\frac{0.6(\Gamma(0.8))^2\left(1-\left(1/\left(10^8(\Gamma(1.8))^2\right)\right)-\left(1/\left(3\times10^7(\Gamma(1.8))^2\right)\right)\right)}{(\Gamma(1.2))^2}\right.\\&+\frac{8.9282\left(1+\left(1/\left(10^8(\Gamma(1.8))^2\right)\right)+\left(1/\left(3\times10^7(\Gamma(1.8))^2\right)\right)\right)}{(\Gamma(1.2))^2},\\&\left.\cdot\frac{1-\left(1/\left(10^2(\Gamma(1.8))^2\right)\right)-\left(1/\left(3\times10^7(\Gamma(1.8))^2\right)\right)}{1.2(\Gamma(0.8))^6}\frac{10^{12}}{6\ln\left(10^2\right)+1}\right[,\end{aligned}\tag{101}$$

and every nonnegative continuous function $g:\mathbb{R}\longrightarrow\mathbb{R}$, there exists $\delta_{\lambda,g}>0$ such that, for each $\mu\in[0,\delta_{\lambda,g}[$, the system (83) has at least three nonnegative weak solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]}|u_1(t)|<10^{-4}$, $\max_{t\in[0,T]}|u_2(t)|<10^4/\sqrt[3]{2}$, and $\max_{t\in[0,T]}|u_3(t)|<10^4$. □

Now, we list some consequences of Theorem 15 as follows.

Theorem 16. *Let f be a nonnegative continuous and nonzero function such that*

$$\lim_{x\longrightarrow 0^+}\frac{f(x)}{|x|^{p-1}}=\lim_{x\longrightarrow+\infty}\frac{f(x)}{|x|^{p-1}}=0,\tag{102}$$

for every $\lambda > \lambda^*$, *where*

$$\lambda^* = \inf \left\{ \frac{(\overline{\varrho_1} + p\bar{M}C(\alpha,\gamma)\overline{\varrho_2})\eta^p}{p\bar{M}TF(\Gamma(2-\alpha)\eta)} : \eta > 0, F(\Gamma(2-\alpha)\eta) > 0 \right\}. \tag{103}$$

Then, there exists

$$\begin{aligned}\bar{\mu}_{\lambda,g} = \min \Bigg\{ \frac{1}{p\bar{M}} \min \Bigg\{ & \frac{\bar{\varrho}_1\theta_1^p - p\bar{M}\lambda TF\left(\sqrt[p]{\theta_1}\right)}{G^{\theta_1}}, \\ & \cdot \frac{\bar{\varrho}_1\theta_4^p - 2p\bar{M}\lambda TF\left(\theta_4/\sqrt[p]{2}\right)}{2G^{\theta_4/\sqrt[p]{2}}}, \frac{\bar{\varrho}_1\theta_4^p - 2p\bar{M}\lambda TF\left(\sqrt[p]{\theta_4}\right)}{2G^{\theta_4}} \Bigg\}, \\ & \cdot \frac{C(\alpha,\gamma)\bar{\varrho}_2\eta^p - \lambda\left((1-2\gamma)TF(\Gamma(2-\alpha)) - TF\left(\sqrt[p]{\theta_1}\right)\right)}{TG_\eta - G^{\theta_1}} \Bigg\},\end{aligned} \tag{104}$$

where θ_1, θ_4, *and* γ *are positive constants with* $\gamma < 1/p$, *such that for each* $\mu \in [0, \bar{\mu}_{\lambda,g}[$, *the problem*

$$\begin{cases} {}_tD_T^\alpha\left(\phi_p({}_0D_t^\alpha u(t))\right) = \lambda F(u) + \mu g(u) + h(u), & t \in [0,T],\ t \neq t_j, \\ \Delta\left({}_tD_T^{\alpha-1}\phi_p({}_0^cD_t^\alpha u_1(t))(t_j)\right) = I_j(u(t_j)), & j = 1, 2, \\ u(0) = u(T) = 0, \end{cases} \tag{105}$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$ *is a nonnegative continuous and nonzero function, has at least two distinct positive weak solutions.*□

Proof. Fix $\lambda > \lambda^*$, put $F(x) = \int_0^x f(\xi)d\xi$ for all $x \in \mathbb{R}$, and let $\eta > 0$ such that $F(\Gamma(2-\alpha)\eta) > 0$ and

$$\lambda > \frac{(\bar{\varrho}_1 + p\bar{M}C(\alpha,\gamma)\bar{\varrho}_2)\eta^p}{p\bar{M}F(\Gamma(2-\alpha)\eta)}. \tag{106}$$

From (102), there is $\theta_1 > 0$ such that $\theta_1 < \min\{\eta, (p\bar{M}C(\alpha,\gamma))^{1/P}\eta\}$ and $F(\sqrt[p]{\theta_1})/\theta_1 < \bar{\varrho}_1/p\bar{M}T\lambda$ and $\theta_4 > 0$ such that $(2p\bar{M}C(\alpha,\lambda)\bar{\varrho}_2/\bar{\varrho}_1)^{1/p}\eta < \theta_4$ and $F(\sqrt[p]{\theta_4})/\theta_4 < \bar{\varrho}_1/2p\bar{M}T\lambda$. Therefore, Theorem 11 ensures the conclusion. □

Finally, by way of example, we point out the following simple consequence of Theorem 16 when $\mu = 0$.

Theorem 17. *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function such that* $xf(x) > 0$ *for all* $x \neq 0$ *and*

$$\lim_{x \longrightarrow 0^+} \frac{f(x)}{|x|^{p-1}} = \lim_{|x| \longrightarrow +\infty} \frac{f(x)}{|x|^{p-1}} = 0. \tag{107}$$

Then, for every $\lambda > \bar{\lambda}$, *where*

$$\begin{aligned}\bar{\lambda} = & \frac{\bar{\varrho}_1 + 2p\bar{M}C(\alpha,\gamma)\bar{\varrho}_2}{p\bar{M}T} \\ & \times \max\left\{ \inf_{\eta>0} \frac{\eta^p}{F(\Gamma(2-\alpha)\eta)}, \inf_{\eta<0} \frac{(-\eta)^p}{F(\Gamma(2-\alpha)\eta)} \right\},\end{aligned} \tag{108}$$

the problem (105), in the case $\mu = 0$, *has at least four distinct nontrivial weak solutions.*

Proof. Setting

$$\begin{aligned} f_1(x) &= \begin{cases} 0, & \text{if } x < 0, \\ f(x), & \text{if } x \geq 0, \end{cases} \\ f_2(x) &= \begin{cases} 0, & \text{if } x < 0, \\ -f(-x), & \text{if } x \geq 0, \end{cases} \end{aligned} \tag{109}$$

and applying Theorem 16 to f_1 and f_2, we have the result. □

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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