What counts? Sources of knowledge in children's acquisition of the successor function

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This work was supported by the National Science Foundation [BCS-1749518, DGE-1321851].

Abstract

Although many US children can count sets by 4 years, it is not until 5½-6 years that they understand how counting relates to number - i.e., that adding 1 to a set necessitates counting up one number. This study examined two knowledge sources that 3½-6-year-olds (*N* = 136) may leverage to acquire this "successor function": (1) mastery of productive rules governing count list generation; and (2) training with "+1" math facts. Both productive counting and "+1" math facts were related to understanding that adding 1 to sets entails counting up one number in the count list; however, even children with robust successor knowledge struggled with its arithmetic expression, suggesting they do not generalize the successor function from "+1" math facts.

Keywords: Count list, Mathematics, Conceptual change, Successor function, Highest count, Decade+Unit rule

Introduction

Although human learners receive linguistic number input that is both finite and varied across contexts and sources, numerate adults nevertheless converge on infinitely productive representations of the natural numbers. By around 6 years of age, many children appear to understand that numbers are infinite, and that any number, *n,* has a successor with the value *n+1*, in keeping with an intuitive understanding of the Peano axioms (Peano, 1889; for alternative formalizations and discussion, see Church, 1932; Decock, 2008; Frege, 1968; Heck, 1995; Wright, 1983; Von Neumann, 1923). In other words, after years of practice counting finite sets, children seem to learn a general property of how the structure of counting relates to number - i.e., that adding one item to a set corresponds to counting up one word in the count list. How do learners impute this limitlessly generative principle from their limited and noisy numerical input, and which sources of knowledge do they leverage in this process? We pursue these questions in the present study by exploring how children use two sources of numerical input — the count list and trained arithmetic operations —to acquire successor function knowledge. Specifically, we test whether children acquire an implicit understanding of the successor function through (a) an analysis of how number words are related (i.e., through mastering the productive rules governing the generation of the count list), or (b) generalizing explicitly trained formal arithmetic operations (i.e., "+1") to all possible numbers.

Symbolic number acquisition is difficult for children and unfolds over many years. Children learn the syntax of number words (Bloom & Wynn, 1997; Ionin & Matushansky, 2019), that they refer to properties of sets, rather than to specific objects (Huang, Spelke, & Snedeker, 2010), and the procedures and principles that govern counting (Fuson, 1988; Wynn, 1990, 1992). Children begin this process early in development, with some studies suggesting sensitivity to both the principles (Ip, Imuta, & Slaughter, 2018) and numerical nature of the count list (Wang & Feigenson, 2019) by 18 months. Around 2 years of age, most US children are able to recite some portion of the count list in an unstructured fashion, similar to the *ABC*s (Fuson, 1988). By about 2½ years, children begin to sequentially acquire the meanings of the first few number words over about 18 months. However, even at this point, children cannot use the count list to accurately label or generate sets larger than 3-4 (Wynn, 1990, 1992). For example, a child who knows the meaning of "two" can generate sets for requests of "one" and "two," but not of "three." It is not until around age 4 that children seem to understand how the count routine can be used to determine a set's cardinality (Sarnecka & Carey, 2008; Wynn, 1990, 1992). At this point, children are typically called "Cardinal Principle" (CP)-knowers, on the premise that they understand how the final word in a count denotes the cardinality of a set (Gelman & Gallistel, 1978).

Despite being able to count sets, many CP knowers appear to lack understanding of *why* counting works, and how the structure of counting is related to number. Some have argued that children acquire the CP through an analogical mapping between the structure of the count list and the meanings of small number words (Carey, 2004; Gentner, 2010; Sarnecka & Carey, 2008; Wynn, 1990, 1992), and that becoming a CP knower amounts to learning that counting up one word in the count list denotes a corresponding +1 increase in the set's cardinality. There is growing evidence, however, that such a mapping is not learned until several years after children acquire the CP (Davidson, Eng, & Barner, 2012; Cheung, Rubenson, & Barner, 2017; Geary, vanMarle, Chu, Rouder, Hoard, & Nugent, 2018; Le Corre, 2014; Spaepen et al., 2018). For example, many young CP knowers cannot determine whether the result of adding 1 item to a set of 5 should be labeled by "six" or "seven." Such failures are evident in the "Unit Task," a paradigm developed by Sarnecka and Carey (2008). In this task, an experimenter hides fish behind an occluder, saying "Look! There are *N* fish in this box!" After hiding the initial set, the experimenter adds 1 additional fish and asks, "Are there *N*+1 or *N*+2 fish now?" Because this task requires reasoning about how the addition of 1 item to an established cardinality is reflected in the count list's structure, some have argued that it requires an implicit understanding of the successor function (Sarnecka & Carey, 2008).

Surprisingly, it takes CP knowers several years to reliably pass the Unit Task, even for well-known numbers that they can easily count, a finding at odds with the hypothesis that acquiring the CP entails mapping between the count list's ordinal structure and cardinality (i.e., that for any number *N*, the next number word always denotes a cardinality of *N*+1). In fact, Davidson et al. (2012) argue that rather than making a generalized induction about the successor function over all numbers, children may instead begin by establishing item-based successor mappings for more familiar numbers around the time they acquire the CP. In line with this, Sarnecka and Carey (2008) found that CP knowers performed with only 67% accuracy on the Unit Task (a task typically used to assess successor knowledge, described in detail below), even for small, familiar numbers (4 and 5). This finding was subsequently replicated for the numbers 5 and 6 (Spaepen et al., 2018). Although CP-knowers in Sarnecka and Carey (2008) outperformed both non-CP knowers and chance (50%), their below-ceiling performance suggests that becoming a CP knower does not guarantee an adult-like understanding of successors, even for small and familiar numbers.

Subsequent work has found that a substantial amount of variability in CP knowers' Unit Task performance is explained by counting skill. Davidson and colleagues (2012) binned CP knowers by how high they could count without making a counting error and found that less proficient counters were at chance on the Unit Task even for numbers well within their count range. More proficient counters, on the other hand, could solve the Unit Task for a greater range of numbers, suggesting a role for counting experience in moving beyond item-based successor relations and imputing a generalized principle. Extending this finding, Cheung and colleagues (2017) found that only extremely competent counters (able to count above about 80 without error) passed the Unit Task for all numbers in their count range (at around 5½ - 6 years of age), and that these children were also more likely to implicate the successor function in explanations of numerical infinity.

Currently, the specific mechanisms through which children acquire this more generalized successor knowledge are not well understood. As reviewed above, there is growing evidence for a link between knowledge of the successor function and counting mastery beyond the CP stage. One account of this relation might be simply that more proficient counters have memorized more of the count list and can perform more operations over this memorized list. Cheung and colleagues (2017) proposed the richer alternative, however, that better counters have not simply memorized the count list but have extracted productive morphosyntactic rules governing the *generation* of that list. For example, base-10 languages like English generate the count list via an additive decade+unit rule which concatenates decade labels like “twenty” or “thirty” with the unit labels from “one” through “nine,” with the units recursively repeated in the same order across every decade. Cheung et al. (2017) proposed that mastering the productive decade+unit structure of the count list may support an induction that numbers themselves are recursively generated via the successor function, as this rule permits children to generate numbers which go well beyond their input (see also Barner, 2017; Hurford, 1987; Rule, Dechter, & Tenenbaum, 2015). This hypothesis predicts that through learning such productive rules, children may infer that every number word has a successor and, by extension, that numbers themselves are unbounded.

While previous work has tested the relation between successor function knowledge and children's rote counting ability, the highest count measure (i.e., the highest number to which children count prior to making an error) is ambiguous as to whether it reflects a purely memorized list or knowledge of productive rules. For example, two children may end their counts on 29, but for two different reasons: One child may do so because that is the only portion of the list they have memorized, while the other may have used a rule to count up from 20, but has not yet memorized the irregular decade-label *thirty.* Given this, much of the previous work using the highest count measure (Cheung et al., 2017; Davidson et al., 2012) cannot speak to whether more proficient counters who demonstrate generalized successor knowledge are children who have learned the productive rules of the count list, or whether they are children who have simply memorized a greater portion of the count list. Recent work, however, has provided some evidence that the origins of successor knowledge may lie in productive, rather than memorized, counting knowledge. First, Schneider and colleagues (2020) found that several measures of counting productivity were robustly predictive of 3- to 6-year-olds' Unit Task performance, and often explained more variance than their highest errorless count. For example, the ability to name the next number from any point in the count list (as opposed to simply counting from "1") was one of the strongest predictors of Unit Task success. Second, in line with the hypothesis that learning these productive rules may fuel inferences about the successor function, Schneider et al. also found lower Unit Task performance among children who speak languages where productive rules are less easily extracted (due to unpredictable morphological and phonological changes; e.g., such as Hindi and Gujarati) in comparison to children who speak languages with more regular counting systems (such as Cantonese, Slovenian, or English). Finally, there is recent evidence that children who understand the decade+unit structure of the count list are more likely to cite the successor function in their justification of statements such as “Numbers go on forever.” (Chu et al., 2020). Together, these results suggest that discovering the productive nature of the count list plays an important role not only in learning how counting up in the count list relates to cardinality (i.e., that adding +1 to a set labeled *n* should be labeled by the next number word in the count list), but also in learning that this successor relation applies recursively, such that numbers never end.

Although productive counting knowledge is one candidate factor in acquiring implicit successor knowledge, children’s numerical input is not limited to the count list. In particular, according to Common Core standards (NCTM, 2000), US children begin to learn formal addition in kindergarten, right around the time they first exhibit generalized successor knowledge. This early addition experience focuses on familiarizing children with the language of equations, mapping them to concrete set operations (similar to the ones involved in the Unit Task), and obtaining addition fluency for a finite set of numbers (e.g., up to 5 in kindergarten, and up to 20 in Grade 1; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). This raises the possibility that children may acquire successor knowledge directly from experience with formal arithmetic training. This alternative route to successor knowledge posits that, as children are exposed to a finite set of “+1” math facts (e.g., *1+1=2, 2+1=3*, etc.), and are taught how the “+1” operation is mapped to set operations, they use this training to infer that the “+1” operation can be applied to any number, and that every number therefore has a successor. That is, as children’s understanding of the set operations involved in addition and their relationship to the “+1” operation increases, they may recognize that the answer to any problem with the form "*n*+1" is the "number after *n*," i.e., the successor function (Baroody, 1995). Thus, in contrast with the hypothesis that children spontaneously notice successor relations on the basis of count list familiarity, this account holds that successor knowledge may emerge from explicit training with addition, which can be measured by children's familiarity with the arithmetic "+1" operation. Thus, this account predicts that children's knowledge of the arithmetic "+1" operation, rather than their counting productivity, should be the strongest predictor of implicit successor knowledge.

Because children's counting proficiency and arithmetic mastery develop in tandem, it is possible that children jointly use these two sources of knowledge to reason about successor relations. For example, while generalized successor knowledge may be driven by extending the "+1" operation to a larger set of numbers than children's initially trained set, the ability to do so may still depend on counting knowledge: Whereas a finite set of math facts might be memorized, deploying these operations for large or unfamiliar numbers surely depends on familiarity with the count list's structure For example, while a child might learn *5+1=6* as part of a routine, it is unlikely that they learn that *75+1=76* in a similar fashion; such computations depend on learning how operations that apply to one decade (e.g., 25+1) extend to all other decades (35+1, 75+1, etc.). On this account, children's knowledge of the productive rules of counting may enable a generalized re-deployment of "+1" operations. Therefore, children's implicit successor knowledge might be equally well-predicted by both their counting productivity and their ability to solve addition operations with "+1".

Compatible with a role for formal training, the operations involved in the Unit Task are directly analogous to "counting on," a procedure sometimes taught to children in an effort to link counting procedures to arithmetic equations (Carpenter & Moser, 1984). Typically, when a 4- or

5-year-old is shown two sets, told the label of the first set (e.g., “Here are five fish”), and then asked to find the total, they count all objects starting with “one,” rather than counting on from “five.” However, 5-year-olds can be trained to “count on" from the first addend, e.g., beginning with “five”, and are especially likely to learn this technique if the first set is covered and can’t be counted. Critically, this technique is challenging for children, and is mastered around 5½ years of age, or right around the time when they are acquiring implicit successor knowledge (Secada, Fuson, & Hall, 1983; see also Carpenter & Moser, 1984). Additionally, counting on is frequently learned either just prior to or concurrent with mastering memorized math facts, and continues to be deployed by many children who are well-trained in arithmetic (Carpenter & Moser, 1984). The algorithmic similarities between the Unit Task and counting on, coupled with their close relation developmentally, suggest that explicit training with arithmetic operations involving “+1” could form the basis for inferring that numbers are generated indefinitely, and that each successive number word denotes a cardinality that is *one* more than its predecessor.

Direct tests of the relation between children’s implicit successor knowledge and their formal addition mastery are limited. Supporting such a link, Hartnett and Gelman (1998) found that 5- to 7-year-olds who were proficient with addition problems involving “+1” and “+2” were more likely to believe that numbers are infinite and that it’s always possible to add 1 to any number. However, this study leaves open the respective roles of counting and arithmetic knowledge, since it failed to establish the sequence of learning, or whether some children performed better across the board simply because they received more exposure to number in general. In another study, Hughes (1981) showed evidence of a dissociation between formal addition and implicit successor knowledge in 3- to 5-year-olds, and that children performed better on a Unit Task analog for sets of 1-8 than when those same numbers were presented in an arithmetic format (e.g., "What does one and two make?"). These results indicate that children's implicit successor knowledge may be unrelated to — and may even precede — their ability to formally express successor relations as arithmetic operations. Hughes (1981) did not assess children's count list knowledge, however, leaving its relation to successor knowledge unclear. Further limiting interpretation of this study, performance was significantly better (approximately 85%) for small numbers (1-3) than for large numbers (approximately 30% for 5-8), raising the question of how children move beyond item-based successor mappings for a subset of small numbers to acquire generalized and productive successor function knowledge, and whether addition training might be involved in this later development. Although such item-based mappings are surely an important component of learning productive rules, they arise much earlier in development, and are only a first step in the learning problem (Cheung et al., 2017; Davidson, Eng, & Barner, 2012).

In the current work, we investigated the relation between children's implicit successor knowledge and two potential factors in the development of this understanding: (1) discovery of the productive rules underlying generation of the count list, and (2) formal arithmetic training on "+1" operations. To do this, we tested children between the ages of 3½ and 6 years on four tasks. We built on prior work (Chu et al., 2020; Schneider et al., 2020) and used two tasks to test children's productive counting knowledge. First, we used the Highest Count task to assess how high children could count without making an error, as well as "productive" counting ability (i.e., the ability to continue counting after an error if given a prompt). We also measured counting productivity using the Next Number task by asking children to "count up" from an arbitrary point in the count list without the benefit of the count routine's momentum. We measured addition mastery by testing children's ability to solve a set of verbally presented arithmetic problems with “+1” (e.g., “What is five plus one?”). Finally, we predicted children’s performance on the Unit Task from these counting productivity and math facts measures. In a set of follow-up analyses, we explored the structure of children’s conceptual and formal arithmetic knowledge across these tasks.

Method

Methods and analyses were pre-registered prior to collection of the full dataset, and before any access to the data (https://osf.io/tfkna?view_only=3eb75cc3444a4187be21b152c3d5a986). All methodological and analytical choices were as pre-registered, unless stated otherwise in text.

Participants

We pre-registered a target *n* of 150 participants pre-exclusions. A power analysis indicated that a sample size of 150 would provide power in excess of .95 in most cases. We recruited 230 children aged 3½ to 6 years from preschools, elementary schools, and the surrounding community. Forty-eight of these children did not complete testing because they were identified as non-CP knowers by an initial screening. Three children were identified as being out of age range, eight were excluded for experimenter error*, two for speech/language delay*, three for experimenter notes to exclude*, thirteen for lacking recordings/data for Highest Count, and two for asking to stop*, leaving 151 participants prior to looking at the data. After looking at the data, we excluded an additional 15 children for missing more than 20% of data (n = 14); and failure to pass training trials (n = 1*). Items with asterisks were not part of our pre-registered protocol, but were in keeping with lab procedures.

After these exclusions, our final analyzable sample included 136 participants identified as CP knowers (n female = 64; n male = 70; n unrecorded sex = 2; M_{age} = 4.92 years; SD_{age} = 0.60 years). While we did not collect other demographic information, our sample was drawn from a population with the following statistics: White (75.5%), Black (5.5%), Asian (12.6%), American Indian or Alaska Native (1.3%), Pacific Islander (0.6%), Multiracial (4.5%) (US Census Bureau).

Stimuli, design, and procedure

Children were tested individually in a small room or area separate from their classroom and received tasks in a fixed order (Give-N, Highest Count, Unit Task, Next Number, and Math Facts).

Give-N. As pre-registered, this task was used as a preliminary screening tool to determine whether children understood the CP. The experimenter provided children with 10 plastic objects (e.g., apples, bananas) and a plastic plate. The experimenter first familiarized the child with the game ("In this game, your job is to put the apples on the plate"), and then asked them to place *N* objects on the plate. The requested set-sizes were 6, 9, 7, and 5, presented once each in a fixed order. After the child indicated they were done, the experimenter asked, "Is that *N*? Can you count and make sure?". If the child wished to make an adjustment, they were allowed to do so. Only children who were able to generate sets for all four requested numbers were classified as CP knowers.

Highest Count. This task measured children's counting ability. In particular, we were interested in how high children could count before making their first error and also whether they could continue counting from this first error when given a prompt. The experimenter began by asking the child, "Can you count as high as you can for me?" As in other work (Almoammer et al., 2013; Barth, Starr, & Sullivan, 2009; Cheung et al., 2017; Fuson, Richards, & Briars, 1982; Davidson et al., 2012; Marušič et al., 2016; Miller & Stigler, 1987), if the child stopped or made a mistake, the experimenter recorded the last number successfully counted to as their "Initial Highest Count." As mentioned in the Introduction, this measure alone may not fully capture children's productive counting knowledge. Many children who are able to successfully count up to (but not beyond) decade transitions have extracted the decade+unit structure of the count list, but are unsure which decade label should come next (Chu et al., 2020; Schneider et al., 2020). That these decade transitions are a more frequent stopping point for children's rote counting than at other places in the count list suggests that Initial Highest Count may underestimate counting productivity for at least some children (Fuson, Richards, & Briars, 1982; Gould, 2017; Siegler & Robinson, 1982). On the other hand, Initial Highest Count could also *overestimate* productivity when rote counts are fully memorized. Compatible with this hypothesis, other work shows that some children are unable to count at all beyond their Initial Highest Count (Chu et al., 2020; Schneider et al., 2020; see also Siegler & Robinson, 1982, for similar findings).

To disentangle rote memorization from productive counting ability, we provided prompts after errors, reasoning that if a child has a productive decade+unit rule then they should be able to incorporate this prompt to continue counting, particularly if the error occurs on a decade transition like 29, 39, etc. When a child made a mistake or stopped counting, the experimenter asked, "What comes after *N*?" If the child did not respond, the experimenter provided a prompt by saying, "Actually, what comes after *N* is *N*+1. Can you keep counting?" The experimenter stopped the task if: the child could not continue; made an error; made more than three errors in a decade; or needed three prompts in a row. Children could receive up to 12 prompts, ensuring that even children who needed prompts at every decade transition could reach the maximum count (120). No child used all 12 prompts; for children who did not spontaneously count to 120 (n = 125), the maximum number of prompts given was 10, and the minimum was 1, with an average of 3.22 (SD = 2.22, Median = 2). Counting data was recorded on a voice recorder, and independently validated by two other researchers.

Prompting yielded two additional measures that were used to assess productivity. The first was children's "Final Highest Count" - i.e., the highest number children reached at the conclusion of the Highest Count task. The second was a binary classification of Resilient or Non-Resilient counter based on the difference between children's Initial and Final Highest Counts. Resilient counters could continue counting at least two decades past any error without making more than three errors within those two decades. The rationale for this criterion, which was pre-registered and based on practices in previous studies (Chu et al., 2020*;* Schneider et al., 2020), was that we expected that children who have extracted the productive rules of the count list should be able to use prompts to recover from their errors and continue to count through a substantial portion of the remaining count list, perhaps only needing prompts on decade transitions. In particular, the two-decade limit was chosen because it offered more compelling evidence of rule-governed knowledge (as opposed to a one-decade limit), while the three-error limit was chosen because it allowed for two errors on decade-change labels (which we expected to pose a challenge for even productive counters) as well as one additional error to allow for errors unrelated to counting knowledge, such as distraction. For example, a child who counted up to 19 and required prompts at 20 and 30, but was able to count to at least 40 would be classified as a Resilient counter.

Unit Task. This task was modeled after Sarnecka and Carey (2008) and was designed to measure children's implicit knowledge of the successor function. The experimenter presented children with a paper pond and explained that they were going to see some fish swimming in the pond. The experimenter placed a clear plastic transparency printed with some number of fish on the pond and said, "Look! There are *N* fish here! *N* fish are swimming under the lily pad." The experimenter then placed a paper lily pad on top of the fish, hiding the set from view, and performed a memory check, asking, “How many fish are under the lily pad?” If the child was unable to remember the starting set, the experimenter repeated the initial presentation until the child successfully remembered the number of fish. After the memory check, the experimenter placed a single fish to the right of the lily pad and said, “Look! Now, are there N+1 or N+2 fish in the pond?” Order of alternatives (N+1 and N+2) was counterbalanced. A *post hoc* test indicated that performance did not significantly differ as a function of alternative order ($t(270) = 1.58$, $p = .11$).

Participants completed two training trials with feedback for sets of 1 and 5, and then 16 test trials (for 6, 7, 15, 20, 32, 34, 46, 51, 57, 60, 62, 73, 81, 84, 93, 95, presented in a pseudo-randomized order) with neutral feedback. These numbers were selected to evenly sample decade labels up to 100, and unit labels from 1-8. We did not include any decade transition items, as pilot testing indicated that these items were associated with chance performance.

Next Number Task. This task, adapted from Hartnett and Gelman (1998), was used as another measure of counting productivity, and required children to count up from arbitrary points in the count list. Thus, this task measured children’s ability to deploy their knowledge of the structure of the count list when not engaged in a rote counting routine (i.e., when not counting up from 1). Naming the next number from a random point in the count list should be an especially strong measure of their productivity, as it requires children to access the decade+unit structure of the count list *without* the benefit of the count list’s momentum. The experimenter introduced the task by saying, “This game is called ‘What Comes Next.’ I’m going to say a number, and then you have to say the number that comes next.” The experimenter prompted the child by saying, “N, what comes next?” Children received a training trial with 1; if they did not pass this trial, they received feedback and support. Any numeric answer was considered a valid response, and a response of “I don’t know” (*n* = 42 of 1154) was coded as incorrect. If the child provided a response less than the initial prompt, the experimenter reminded them that they were to provide the *next* number. Children only received one such reminder. Participants received eight trials in a pseudo-randomized order with neutral feedback. Three of the items (7, 62, and 95) were repeated from the Unit Task, while the remaining five (24, 26, 30, 71, and 83) were novel.

Our pre-registered productivity measure using this task was the highest number for which children generated a correct response, as we reasoned that children who had extracted the productive rules of the count list should reach higher numbers in this task. However, we amended this measure shortly after data collection began, as we noted it was possible for children to generate a very high next number on this task purely by chance, despite having very low accuracy overall. This modified measure was Highest Contiguous Next Number, the highest number for which children were able to correctly respond in the task, provided that all the previous items were correct. For example, a child who responded correctly for 7, 24, 26, and 62, but failed on 30 and 71 would have a Highest Contiguous Next Number of 26.

Math Facts Task. This task measured children’s mastery of the “+1” operation. The experimenter did not indicate that this task was about mathematics, but asked only, “What is *N plus* 1?” We used a free response format in this task because it is most similar to how children might encounter arithmetic equations in the classroom. Children received a training trial on “*1* plus *1*.” While almost all children could correctly respond to this question (*n* = 128 out of 136), the experimenter provided feedback if the child was unable to answer. After the training trial, participants completed eight trials in a pseudo-randomized order without feedback. Four of the numbers in this task (20, 32, 57, 93) overlapped with the Unit Task, while the remaining four (5, 21, 64, 86) were novel.

Indefinite Next Number Task. Many languages feature so-called "indefinite" numbers, like "umpteen," "zillion," etc. that aren't associated with a definite cardinality. This task was included as an exploratory test of whether children's knowledge was sufficiently general that it could be applied to such numbers, or at least to very large and less familiar numbers. In the task, children were asked to generate the next number in response to either very large numbers (1,006, 1,057) or indefinite numbers (zillion 41, zillion 73). We reasoned that if children truly understand the productive rules governing number word generation, they should be able to generate the successor of *any* number word, regardless of its familiarity. At the same time, we expected that children's ability to express this knowledge might be tempered by other non-linguistic abilities, such as working memory. Because children could struggle with this task for reasons unrelated to productivity that this study was not designed to test, it was pre-registered as an exploratory measure and not associated with any specific predictions.

Results

Predictors of Unit Task performance

Our primary question was whether children acquire knowledge of the successor function through an induction made over count list familiarity, through generalizing trained arithmetic (i.e., "+1"), or via a combination of both mechanisms. To address this, we constructed several generalized linear mixed effects models (GLMMs) predicting Unit Task performance from (1) Initial Highest Count; (2) Final Highest Count; (3) Counting Resilience; (4) Highest Contiguous Next Number; and (5) mean Math Facts performance. We first tested whether productive counting knowledge and formal addition mastery were related to Unit Task performance in two independent models, and then combined significant predictors in a single large model.

To construct our counting productivity model, we first tested whether children's Initial Highest Count, Final Highest Count, Counting Resilience, or Highest Contiguous Next Number individually predicted Unit Task performance with a Likelihood Ratio Test against a base model without that productivity term. All GLMMs were fit in R using the 'lme4' package (Bates, Martin, Maechler, Bolker, & Walker, 2015). The full model specification was: Correct response ~ [Initial Highest Count/Final Highest Count/Resilience/Highest Contiguous Next Number/Mean Math Facts] + Item within or outside Initial Highest Count + Age + (1|Subject) + (1|Task Item). We deviated from our pre-registration to include a random effect of task item, reasoning that performance for some items might be more difficult than for others.

Likelihood Ratio Tests indicated that all four productivity measures significantly improved the fit in comparison to the base (all *p*s < .001, Figure 1). Because these predictors are not mutually exclusive, we further tested whether they explained unique or overlapping variance by hierarchically adding each term to a single model in order of increasing AIC, and conducting another Likelihood Ratio Test with the addition of each term. The base for this large model contained Highest Contiguous Next Number, which was associated with the lowest AIC. The addition of Final Highest Count significantly improved model fit ($\chi^2_{(1)} = 10.48$, $p = .001$), but neither Initial Highest Count ($\chi^2_{(1)} = 1.43$, $p = .23$) nor Resilience ($\chi^2_{(1)} = 2.32$, $p = .13$) explained additional variance. Thus, like other work (Schneider et al., 2020), we found that although multiple measures of count list mastery were related to implicit successor knowledge, measures which better disambiguate between rote and productive count list knowledge — here, Highest Contiguous Next Number and Final Highest Count — emerge as the strongest predictors.

We next constructed a separate model to test whether children's mastery of "+1" Math Facts operations was predictive of Unit Task performance. A Likelihood Ratio Test indicated that the addition of mean Math Facts performance to the base significantly improved the fit ($\chi^2_{(1)}$ = 52.12, $p < .001$; Figure 1).

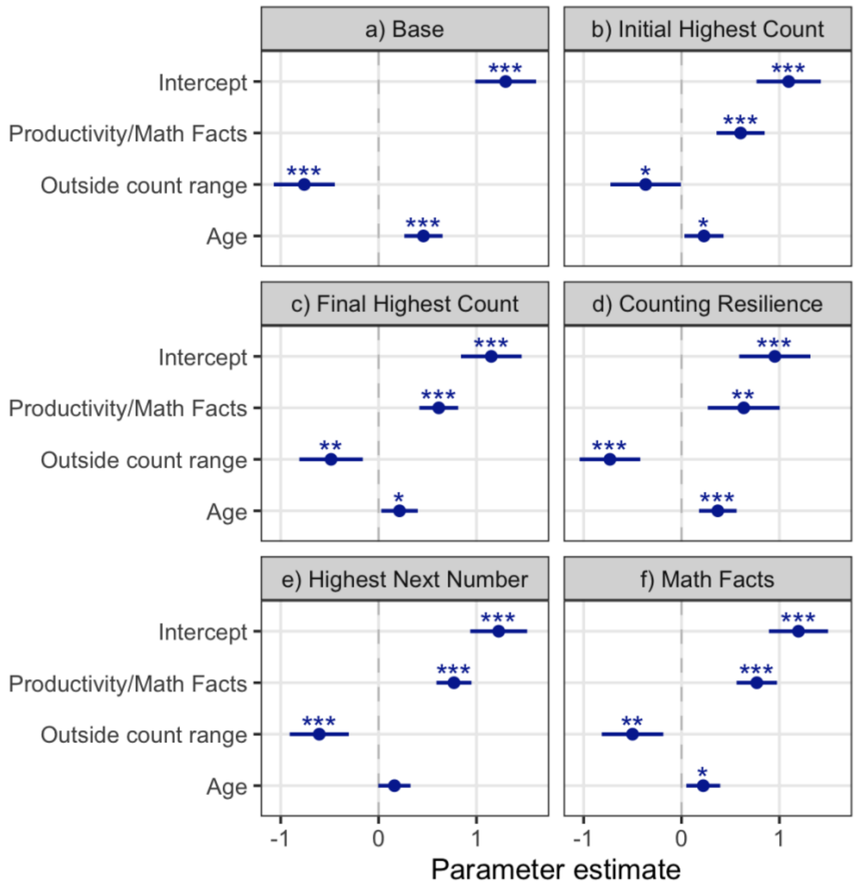

Figure 1. Parameter estimates for individual models predicting Unit Task performance from a) Count range and age (base model); b) Initial Highest Count; c) Final Highest Count; d) Counting Resilience; e) Highest Contiguous Next Number; and f) Mean Math Facts performance. Error bars represent 95% confidence intervals. *$p < .05$; **$p < .01$; ***$p < .001$.

Finally, to test whether arithmetic training with "+1" explained unique variance in Unit Task performance beyond counting productivity, we added mean Math Facts performance to the model containing both Final Highest Count and Highest Contiguous Next Number. The addition of mean Math Facts performance significantly improved model fit ($\chi^2_{(1)}$ = 14.92, $p < .001$), suggesting that Unit Task performance is related to both productive count list knowledge and proficiency solving formal addition problems with "+1" (Figure 2). While Final Highest Count was no longer significant in this final model ($\beta = 0.17$, $p = .08$), Highest Contiguous Next

Number remained significant, and was a stronger predictor (β = 0.53, p < .001) than Math Facts performance (β = 0.42, p < .001). Additionally, this final model showed less accurate performance for items outside children's Initial Highest Count (β = -0.36, p = .03), with no effect of age.

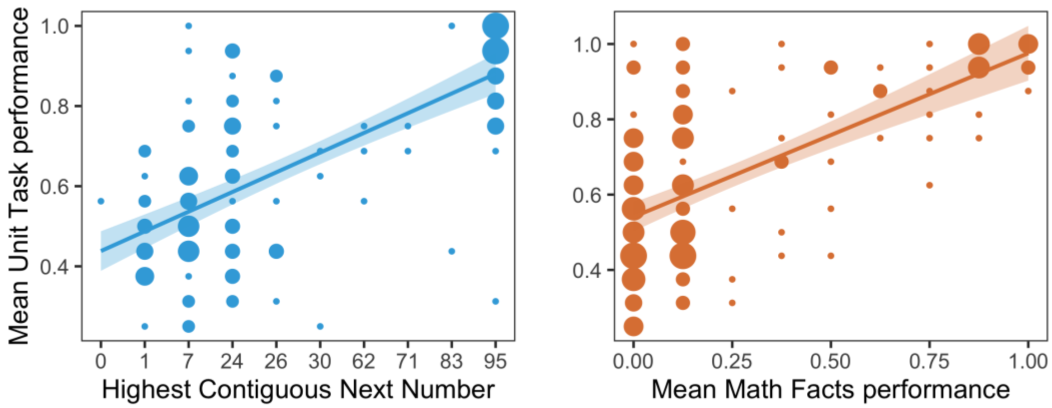

Figure 2. Scatterplots of relation between mean Unit Task performance and Highest Contiguous Next Number (left) and mean Math Facts performance (right). The size of each point represents the frequency of a value. Smoothed lines indicate linear fit, and shaded regions indicate 95% confidence intervals.

While we find that Unit task performance is correlated with both productive counting and knowledge of "+1" math facts, correlational data cannot be taken as straightforward evidence for causation. Very generally, there are many possible reasons for positive correlations: One possibility is that some children simply receive more training on number related activities of all types (i.e., performance on these tasks is independently affected by number rich environments). Another possibility is that there are domain-general factors (e.g., working memory) that limit performance on number-related tasks, leading to correlations between them. Finally, it is possible that there are indirect causal relations between certain tasks - e.g., that Math Facts and the Unit Task are each constrained by a third variable like counting knowledge, despite not having direct causal interaction with one another (since both require knowledge of number words and how they are ordered). Given these considerations, we reasoned that if an event *B* follows an event *A*, then it's unlikely that *B* causes *A* developmentally. Given these considerations, we therefore asked whether children's performance on the Unit, Next Number, and Math Facts tasks emerged systematically earlier or later relative to the other tasks.

A preliminary comparison found that performance on the Math Facts task was substantially lower overall (M = .27, SD = 0.44) than performance on either the Unit Task (M = 0.66, SD = 0.48) or Next Number (M = 0.67, SD = 0.47). This is important, since if Math Facts knowledge were a prerequisite to Unit Task knowledge, we would expect Math Facts performance to be equal to or better than Unit Task performance. One possible concern, however, is that the Math Facts and Unit Tasks featured different response formats that might affect overall performance; whereas Math Facts was open-ended, the Unit Task was a two alternative forced choice task. Critically, however, we found no main effect of response format: although the Next Number task also had an open-ended response format, it did not differ from the Unit Task, whereas both differed from Math Facts (Figure 3). Also militating against the idea that Unit Task performance was better than Math Facts because the task was easier, only the Unit Task required reasoning about both relations between numbers in the count list and how these relations correspond to set operations (e.g., how counting up one word relates to adding one object to a set). Further, the same pattern of results on these three tasks was found in a small pilot study (n = 9) in which each task featured an open-ended response format. In that pilot study we found that Math Facts performance was still significantly lower (M = .51) than both the Unit (M = .80) and Next Number tasks (M = .77), again suggesting that the lower Math Facts performance we observed here was not an artifact of response format. Together, these results indicate that children are capable of performing the set operations involved in the Unit Task and using the count list's structure to generate number words before they can solve arithmetic "+1" operations involving these same number words.

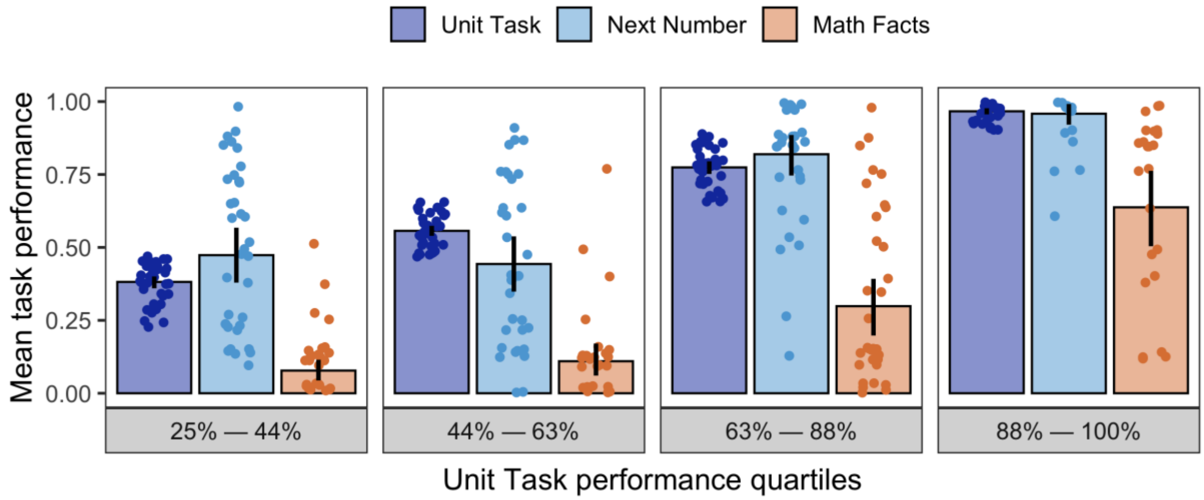

Figure 3. Mean task performance for Unit, Next Number, and Math Facts tasks, grouped by Unit Task performance. Points represent individual participants' mean performance and are jittered slightly to avoid overplotting. Error bars indicate 95% confidence intervals, computed by nonparametric bootstrap.

While a comparison of overall accuracy indicates lower Math Facts performance in relation to the Unit Task, the critical test of whether children use their arithmetic training with "+1" to infer the successor function, however, is whether participants who demonstrate generalized successor knowledge (i.e., children who are at or close to ceiling Unit Task performance) are also at ceiling on the Math Facts task. This logic of this pre-registered comparison is that, if learning generalized knowledge of the successor function results from an inductive inference based on arithmetic training with "+1", then any child who exhibits such knowledge in the Unit Task should also be at ceiling in the Math Facts task (given that the same or similar numbers were tested in both tasks). To test this, we investigated whether the difference between the Unit and Math Facts tasks was also present in children at ceiling on the Unit Task. Using data from only participants in the top quartile of the Unit Task (at least 87.5% mean performance, n = 30), we constructed a GLMM predicting accuracy from Task, controlling for whether the queried item was within a child's Initial Highest Count and the child's age, with random effects of participant and task item. This analysis revealed significantly lower Math

Facts accuracy (64% mean performance) in comparison to the Unit Task (β = -3.28, p < .001), with contrasts indicating that Math Facts performance was significantly poorer than both Unit Task and Next Number (β = -3.15, p < .001), but that performance did not differ between the Unit Task and Next Number (β = 0.27, p =.53), as shown in Figure 3. Note that if we conduct the (exploratory) reverse comparison - whether children at ceiling on Math Facts (at least 87.5% mean performance) were worse on the Unit Task - we do not find the same result. Instead, we find that 6/7 of these children were also at ceiling on the Unit Task (though due to the very small number of children at ceiling on Math Facts — a telling result in itself — no statistical analysis was possible here).

Thus, while we found a positive correlation between Math Facts and Unit Task performance that was compatible with multiple causal hypotheses, follow-up analyses revealed a more nuanced picture of the relation between these tasks: performance on Math Facts was significantly lower than either the Unit Task or Next Number for all children, and that this was not explained by task format, or limited to only the children with weak successor knowledge. This difference in accuracy was present in children who were at ceiling in the Unit Task, and who would have been most likely to also have generalized “+1” knowledge. Together, these results suggest that although children who perform well on Math Facts are more likely to also perform well on the Unit Task in comparison to children who struggle with Math Facts (i.e., the tasks are correlated), children’s knowledge of arithmetic +1 operations appears to develop later overall, indicating that arithmetic training is likely not causally implicated in acquiring implicit successor knowledge.

Investigating the dissociation between addition concepts and formal arithmetic

In the above analyses, we found that many children who were able to perform the operations in the Unit Task did not understand how to compute those same operations when described using arithmetic language — that is, many children could report that, e.g., adding 1 fish to a set of 5 results in a set of six fish, but were nevertheless unable to report that “five plus one equals six”. This suggests that prior to learning specific math facts, like 20+1=21, many children have fairly robust conceptual representations of the operations these equations represent, consistent with other work showing that children’s knowledge of the set-operations associated with arithmetic often precedes learning their associated formal expression and procedures (Carpenter & Moser, 1984; Hughes, 1981; Huttenlocher, Jordan, & Levine, 1994). Given this, children’s primary challenge at this stage of mathematical learning may not be conceptual, but instead a problem of learning how to express existing conceptual relationships using formal language. Such a lag might take two distinct forms. On one hand, children may simply lack knowledge of the linguistic or symbolic representations of addition equations, i.e., *“X plus Y equals Z” or “X + Y = Z”*. On this view, once children learn one or two equations, they should be able to use their existing conceptual addition knowledge to infer that if 5+1=6, then for any number *N*, the equation *N*+1 can be solved by identifying the number that comes after *N* in the count list. Alternatively, however, it may be that children *are* familiar with a subset of math fact equations, and simply haven’t realized that math facts represent the same content as counting - i.e., that the “+1” operation is equivalent to the successor function in its conceptual content. On this view, we should expect to find some children who succeed at a subset of math facts, but still show a significant advantage for Unit Task performance over Math Facts performance for larger numbers - e.g., compatible with a failure to make a generalized connection between the two types of knowledge. We pursued these two possibilities by exploring the disconnect between children's Unit Task and Math Facts performance in a *post hoc* analysis.

First, compatible with the second hypothesis, almost all children (94%) correctly and spontaneously answered the Math Facts warm-up ("Do you know what 1 *plus* 1 is?"). As quantities increased, however, performance quickly decreased, suggesting that although children are familiar with the "*X plus Y equals Z"* template and know how to respond to it for a specific subset of cases, their knowledge is item-specific, and has not been integrated with their existing, stronger knowledge of successor relations (as indicated by their Unit Task performance). To further explore this relation, we used a GLMM to test for an interaction between number magnitude and task (Unit Task and Math Facts) with the formula: Correct ~ Number magnitude*Task + Item within or outside Initial Highest Count + Age + (1|SID). Because only four of the Math Facts items overlapped with the Unit Task (20, 32, 57, and 93), we matched the remaining items (5, 64, and 86) on magnitude with the closest Unit Task item (6, 62, and 84), averaging each number pair to create a "magnitude match." For example, we used a magnitude match of 5.5 to compare children's performance on 5 in Math Facts and 6 in the Unit Task.

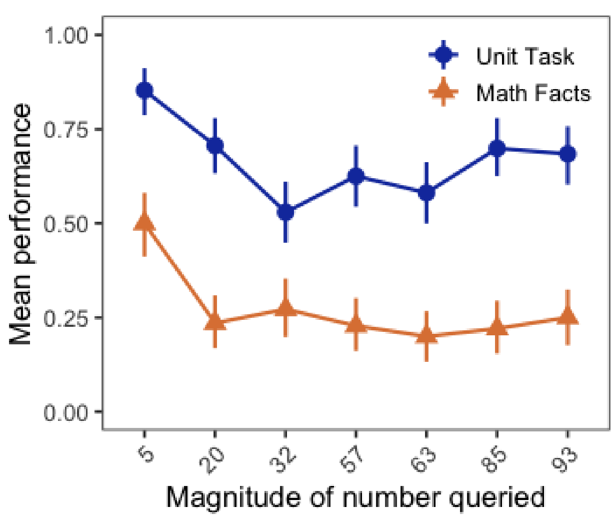

Figure 4. Mean performance for the Unit Task and Math Facts for the magnitude of each queried item. Error bars indicate 95% confidence intervals, computed by nonparametric bootstrap.

This analysis found a significant interaction between magnitude and task ($\chi^2_{(1)} = 6.30$, p = .01), such that performance for Math Facts significantly worsened as magnitude increased (β = -0.30, p = .01) in comparison to the Unit Task (Figure 4). This result suggests that, despite familiarity with the language of equations, and despite having conceptual knowledge of the set operations they ultimately represent, many children have not connected these two types of knowledge. That is, while many children in our sample were able to state that the result of adding 1 to a set of 20 resulted in a set of 21, they could not solve the equation "20 *plus* 1," even if they had correctly solved "5 *plus* 1." A visual inspection of the data (see Figure 4) suggests that this effect is driven by a rapid decline in performance after our "5 plus 1" trial, likely because this item is more likely to be encountered as part of children's rote-trained set (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Although our comparisons for smaller magnitude numbers were limited, the finding that approximately half of the children in our sample were familiar enough with the language of equations to correctly solve 5+1=6, but were unable to do so for larger numbers, suggests that children may initially rote memorize math facts as item-specific expressions without mapping them to their existing knowledge of successor relations.

Exploratory: Indefinite Next Number Task

The preceding analyses revealed a robust relation between children's productive counting ability and their implicit successor knowledge. Additionally, we found that although mastery of formal math facts was related to successor knowledge, it is unlikely that children leverage knowledge of the "+1" operation to induce the successor function. Rather, our Unit Task models revealed that children's ability to count up from an arbitrary point in the count list in the Next Number task was most closely related to their Unit Task performance.

One potential limitation, however, is that all items on the Next Number task could be plausibly contained within a child's memorized count range, and could still reflect a rote-learned count list. While we reasoned that the Next Number task should be much more difficult if children have a purely memorized, rather than productive, count list representation, a much stronger test of this hypothesis is to probe unfamiliar numbers that could not possibly be rote learned. We addressed this issue using the Indefinite Number Task, which included two very large numbers (1,006 and 1,057) and two nonspecific, but potentially familiar numbers (zillion 41 and zillion 73). In this exploratory analysis, we tested whether Unit Task performance was related to the ability to productively generate successors on the Indefinite Number Task. These analyses included 101 participants (M_{age} = 4.93, SD_{age} = 0.63), as this task was included as an exploratory measure and was not completed by all participants.

Overall, performance on this task was quite low (*M* = 0.22, *SD* = 0.41), indicating that identifying the successor of these extremely large numbers was much more difficult than for more familiar numbers. We first explored whether children's ability to generate successors for the largest numbers queried in the Unit Task (81, 84, 93, and 95) was predictive of their Indefinite Next Number performance by constructing a GLMM with the formula: Indefinite Next Number Correct Response ~ Mean Unit Task performance for 81, 84, 93, and 95 + Age + (1|Subject) + (1|Indefinite task item). This model revealed that Indefinite Next Number performance was significantly related to success on the largest Unit Task items (β = 1.98, p < .001) and age (β = 1.43, p = .004). In fact, children who correctly passed the Unit Task for all four of these items had substantially higher Indefinite Next Number accuracy (*M* = 43%) in comparison to children who passed three (*M* = 17%) or two (*M* = 7%) of these items.

Next, we tested whether the ability to generate Indefinite successors was related to overall Unit Task performance by adding it to a GLMM predicting a correct Unit Task response from whether the queried item was within the child's count range and their age, with random effects of subject and Unit Task item. A Likelihood Ratio Test indicated that Indefinite Next Number significantly improved model fit ($\chi^2_{(1)} = 49.22$, $p < .001$), with greater performance associated with higher Unit Task accuracy ($\beta = 0.87$, $p < .001$).

Finally, we tested whether Indefinite Next Number performance accounted for unique variance beyond both the productivity measures identified in our primary analyses (Final Highest Count and Highest Contiguous Next Number) and Math Facts performance. We did this by adding mean Indefinite Next Number performance to our final Unit Task model, controlling for the effects of these three predictors, with model formula: Correct ~ Mean Indefinite Next Number performance + Mean Math Facts performance + Final Highest Count + Highest Contiguous Next Number + Trial within or outside participant's Initial Highest Count + Age + (1|Subject) + (1|Task item). Interestingly, as shown in Figure 5, this measure not only significantly improved the fit of our final Unit Task model ($\chi^2_{(1)} = 11.19$, $p < .001$), but was also as strong a predictor ($\beta = 0.42$) as Highest Contiguous Next Number ($\beta = 0.42$), and was a stronger predictor than Math Facts ($\beta = 0.24$). Thus, despite low overall performance on this task, children who can use their productive counting knowledge to generate the successor of *any* number, even unfamiliar ones, are more likely to demonstrate implicit successor knowledge.

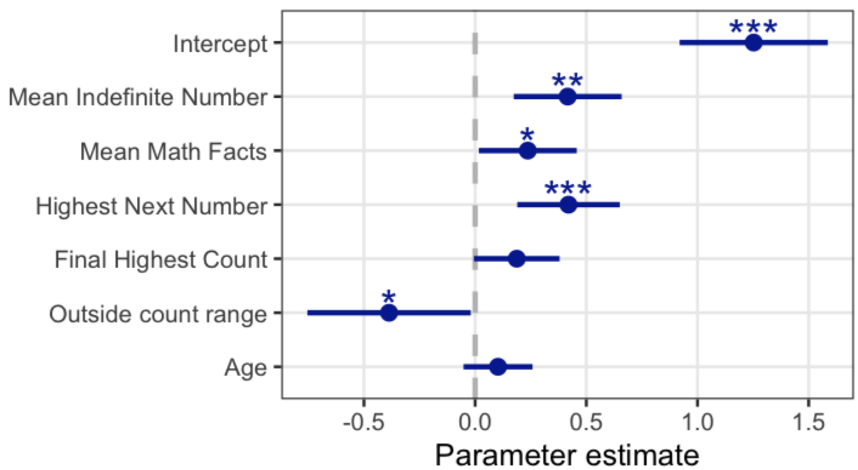

Figure 5. Parameter estimates of final large Unit Task model including mean Indefinite Next Number performance. Error bars represent 95% confidence intervals. *$p < .05$; **$p < .01$; ***$p < .001$.

General Discussion

Over several years, children aggregate several distinct forms of numerical information to acquire a mature understanding of number. Here, we explored how children leverage two potential sources of numerical input to acquire implicit knowledge of the successor function, a logical principle stating that every number n has a successor, n+1. Specifically, we asked how successor function knowledge is related to learning the productive rules of their count list, and trained arithmetic operations ("+1"). While children's knowledge of "+1" operations was correlated with Unit Task performance (our measure of implicit successor function knowledge), we found significantly lower performance for these formal addition operations. This suggests that it is unlikely that children draw upon arithmetic training in acquiring implicit successor knowledge, and that instead it is more likely that successor knowledge plays a role in learning addition. In contrast, we found strong evidence that knowledge of the count list's structure — in particular, the productive rules underlying number word generation — was closely related to Unit Task performance, and that this predictive relation held across multiple measures of productivity.

We found several pieces of evidence that this productive counting knowledge may fuel children's inferences about the successor function. First, one of the strongest predictors of Unit Task performance was the Next Number task, an especially strong measure of productivity which requires knowledge of the decade+unit structure of the count list to generate the successor to a random number. Next Number offers a strong assessment of children's productivity in comparison to rote counting because it isolates productive counting knowledge both from rote counting's demands (attention, memory, and endurance) and momentum (Hartnett & Gelman, 1998; Siegler & Robinson, 1982). Second, in a task that asked children to count as high as they could, we found that children who could recover from counting errors ("Resilient" counters) demonstrated more generalized successor knowledge than those who could not. Further, our analyses indicated that children's Final Highest Count, which can potentially disambiguate between memorized and productive counting, was a better predictor of Unit Task performance than their Initial Highest Count. Finally, one of the strongest predictors of Unit Task performance was children's ability to generate the next number for entirely unfamiliar numbers, like "a zillion 42." Taken together, these results indicate that children who have extracted the productive rules of their count list may be in a position to make an induction about the relation between the successor function and the generation of these number words.

This strong relation between counting productivity and successor function knowledge is important to understanding how children may acquire a form of this logical principle. Previous work (Cheung et al., 2017; Davidson et al., 2012; Hartnett & Gelman, 1998) has found that counting ability is closely linked to successor knowledge; both Cheung et al. (2017) and Davidson et al. (2012) found that many less proficient counters possess localized successor knowledge, and could implement the successor function for only some numbers within their count range. Cheung and colleagues found that this localized network grew along with children's counting proficiency, slowly solidifying into a generalized principle. The mechanism underlying this relation was not made clear by this work, however; one possibility was that children who could count higher had simply memorized more of the count routine and used that memorized list to succeed on the Unit Task. The current work, coupled with other subsequent studies on this topic (Chu et al., 2020; Schneider et al., 2020), suggests that this is not the case, and that it is children's mastery of the productive morphosyntactic rules governing number word generation, and not merely reliance on a memorized list, that is most predictive of their successor knowledge. Thus, it is possible that one of the first steps in recognizing that numbers are endless is learning the generative linguistic machinery that makes number *words* endless.

In addition to productive counting knowledge, we also found that Unit Task performance was correlated with children's arithmetic training, as measured by their ability to solve addition equations with "+1". Despite this relation, we found several indications that children's isolation and extension of this arithmetic operation was likely not implicated in the emergence of implicit successor knowledge. First, we found that for all children, knowledge of how to productively generate number words was much more closely related to Unit Task performance than the ability to solve addition equations with the same or similar numbers. If children acquire the successor function through their training with arithmetic and the "+1" operation, we should expect no difference in performance between these two tasks. Second, this difference between Math Facts and Unit Task performance was present even for children who performed at ceiling on the Unit Task; this is notable because these are precisely the children that we would expect to have the most generalized understanding of the "+1" operation if it were the means through which they were inferring the successor function. Finally, we found that successfully implementing the successor function on the Unit Task for a given number was not related to whether a child could solve an addition equation containing the same or a similar number, indicating that many children may not even understand what the "+1" operation actually encodes.

These results are consistent with Hughes' (1981) proposal that children's mastery of formal arithmetic code may constrain early mathematics achievement, but not because children are unfamiliar with this language (as Hughes suggested), but because they have not mapped this language to the appropriate concepts. Interestingly, our results indicate that many children are familiar with the general template of addition equations with "+1" (94% of children could solve "1 plus 1," and 50% could solve "5 plus 1") but may initially learn these operations in an item-based fashion before connecting them to their underlying conceptual content. Supporting this hypothesis, we found that Unit Task and Math Facts performance was more closely related for smaller and more familiar numbers than larger ones. For example, approximately half of the children who successfully answered, "What's 5 *plus* 1?" could not answer "What's 20 *plus* 1?", despite successfully performing that operation in the Unit Task. These results are compatible with previous findings that children rely on more intuitive methods for solving arithmetic problems in lieu of formal arithmetic, even after several years of formal addition training (Carpenter & Moser, 1984), and with the suggestion that children's conceptual understanding of arithmetic operations may be in place prior to mastering the language of mathematics (Carpenter & Moser, 1984; Hughes, 1981; Huttenlocher, Jordan, & Levine, 1994). Our data further suggest that even children who have the template of formal addition operations still struggle to map this language to the appropriate underlying operations.

There are several limitations in the current work which provide directions for future study. First, as we noted in our discussion of the results, it is possible that the observed difference between the Unit Task and Math Facts tasks is in part affected by a higher baseline level of accuracy for chance performance (50%) in the Unit Task. However, against this possibility, the current study did not find that children's performance was affected by having to generate a free response on the Next Number task, suggesting that children were capable of correctly generating their own alternatives. Also, we found lower Math Facts performance in comparison to both the Unit and Next Number tasks in a small pilot sample which used free response for all tasks. Still, future work should equate methods of response across these tasks to provide further clarity regarding the differences between these tasks.

Second, although we found that children succeed on the Unit Task well before Math Facts, it is still possible that they draw upon their formal addition training later in the learning process, when they begin to generalize the successor function to larger numbers and reason about it explicitly. While Cheung and colleagues (2017) found that children who were at ceiling in the Unit Task were more likely to understand that numbers went on forever by virtue of the successor function, Hartnett and Gelman (1998) argued that children acquire this knowledge only through mastering addition operations. Meanwhile, in the current work, we explored only children's understanding that the addition of one item to a set corresponds to a "+1" increase in the count list, but not whether children who succeeded on the Unit Task were able to either (1) explicitly articulate this principle, and (2) understand that this principle renders the natural numbers endless. One possibility, in line with Cheung et al.'s (2017) proposal, is that children may be in a position to infer the fully generalized successor function purely on the basis of the count list's structure; through recognizing that the successor to any given number is always generated through implementing the successor function, children may eventually recognize that this process applies to all possible numbers, and can be infinitely implemented. Another possibility, however, is that children may use all the numerical knowledge they have at their disposal; while the structure of the count list may give children the basic machinery of the successor function, they may also draw upon their addition knowledge to formalize this principle and conclude that this “+1” operation can be repeated infinitely. Our current data do not differentiate between these two alternatives since we did not explicitly probe beliefs about infinity. Future work should explore this question, and whether children generalize their successor knowledge on the basis of productive counting knowledge alone, or whether later mastery of formal arithmetic might also play a role.

Acquiring counting knowledge that respects the successor function is one critical step in developing a full understanding of the natural numbers (Izard, Pica, Spelke, & Dehaene, 2008), and potentially provides the framework for the discovery of numerical infinity (Cheung et al., 2017; Chu et al., 2020). Discovering the mechanisms which enable children go beyond their finite numerical input in learning this infinitely productive logical principle is therefore key not only to furthering our understanding of children’s number acquisition, but also to identifying areas of instruction which may benefit children in the pursuit of this knowledge. Our finding that children’s mastery of the count list and its productive structure is related to acquiring implicit successor knowledge suggests that interventions aimed not just at achieving counting fluency, but on extracting the productive morphosyntax of counting, may help children discover how the successor function is implicated in the count list. Such an intervention may be effective not only for children in the US, but also for children learning to count in languages where such productive counting rules are less easily discovered (Schneider et al., 2020).

In conclusion, our current work, coupled with other recent studies in the literature (Chu et al., 2020; Schneider et al., 2020), supports the idea that the origins of children’s successor knowledge may lie in their understanding of the productive morphosyntactic rules of their language's count list, and that this knowledge precedes mastery of its formal arithmetic expression. Children who have a strong grasp of the productive rules underlying the generation of the count list may be in a strong position to make an inference about the relation between these syntactic rules and the successor function and seem to do so independently of formal addition training with "+1."

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What Counts? Sources of Knowledge in Children's Acquisition of the Successor Function

Supplementary Online Material

Table of Contents

1. Highest Count

We tested children's counting ability using the Highest Count task. Similar to other work (Almoammer et al., 2013; Barth, Starr, & Sullivan, 2009; Cheung et al., 2017; Fuson, Richards, & Briars, 1982; Davidson et al., 2012; Marušič et al., 2016; Miller & Stigler, 1987), we classified the highest number to which children could count prior to making an error as their "Initial Highest Count." This measure is ambiguous with respect to whether it reflects a rote-memorized list or productive counting knowledge; for example, this measure does not yield a clear signal on whether two children who stop counting at "29" have done so for the same reason, making it an imperfect measure of counting productivity.

To provide a clearer signal of rote vs. productive counting ability, we provided children with prompts when they made counting errors (e.g., "Actually, what comes after 29 is 30. Can you keep counting?"). The logic of providing these prompts was that, if children understand the decade+unit structure of the count list but have made an error due to an irregular decade label, then they should be able to incorporate this prompt into the productive decade+unit template to continue counting substantially past their error. The highest number to which children could count with the aid of these prompts is their "Final Highest Count."

Finally, we reasoned that the difference between children's Initial and Final Highest Counts should also serve as a metric of counting productivity, under the assumption that children who are productive counters should use prompts to progress through a larger portion of the count list than non-productive counters. Thus, we classified any child who was able to count at least two decades past an error with no more than three errors within those two decades as a “Resilient” counter, while “Non-Resilient” counters could not meet this threshold. The logic of this pre-registered criterion was that the ability to count two decades past an error (with potentially the only additional errors being incurred at decade transitions) demonstrates some rule-governed knowledge (as opposed to a one-decade criterion), while accommodating children who were able to count up to 100, where additional embedding adds perhaps an additional level of irregularity.

Of the 125 children who were not able to count to the maximum number tested in Highest Count (120), 94 were able to count beyond their Initial Highest Count (*M* = 34, *SD* = 30, Median = 21). That the majority of children were able to recover (even minimally) from a counting error strongly suggests that Initial Highest Count may not fully capture productive counting knowledge. Summary statistics of Initial and Final Highest Counts are shown in Figure 1 and detailed in Table 1.

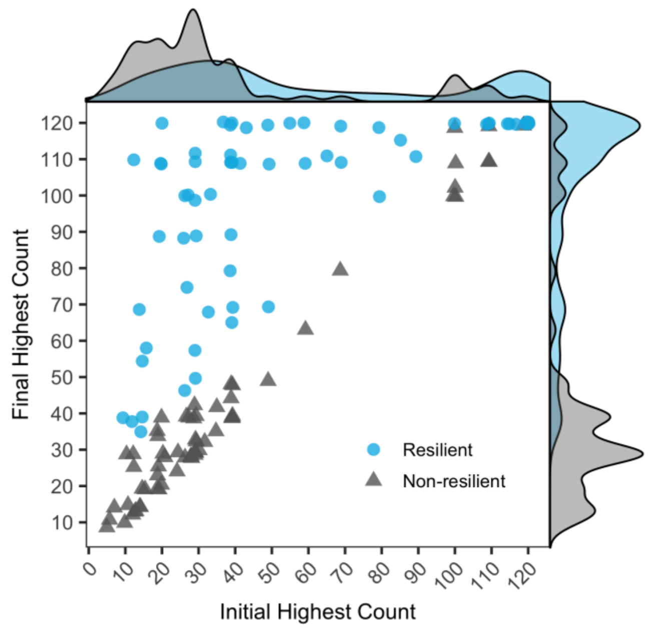

Figure 1. Scatterplot of Initial and Final Highest Counts grouped by Counting Resilience. Points are jittered slightly to avoid overplotting. Density plots indicate the distribution of Initial (top) and Final (right) Highest Count by Resilience.

	n	*M* IHC (*SD*)	Median IHC	*M* FHC (*SD*)	Median FHC
Overall	136	48 (37)	33	71 (41)	69
Resilient	70	59 (39)	39	100 (26)	110
Non-Resilient	66	36 (30)	29	41 (30)	30

Table 1. Summary of Initial (IHC) and Final Highest Counts (FHC), overall and grouped by Counting Resilience. Counts are rounded.

2. Task Performance by Counting Resilience

In these analyses, we explored differences in performance across all three tasks (Unit Task, Next Number, and Math Facts) between Resilient and Non-Resilient Counters. In our primary analyses, we found that children's counting productivity was significantly related to their Unit Task performance. Here, we use the broad Resilient and Non-Resilient classification to explore the relation between productivity and other measures of numerical knowledge. To do this, we built a GLMM predicting a correct response on all three tasks (Unit, Next Number, and

Math Facts) from Counting Resilience, task, whether the item was within a participant's Initial Highest Count range, and age, with a random effect of subject. This model indicated significant main effects of Task ($\chi^2_{(2)} = 535.02$, $p < .001$) and Counting Resilience ($\chi^2_{(1)} = 16.11$, $p < .001$), with Resilient Counters significantly out-performing Non-Resilient Counters ($\beta = 0.83$, $p < .001$). Reflecting the results of our primary analyses, *post hoc* contrasts indicated that Math Facts performance was significantly lower than either Unit or Next Number performance ($\beta = -2.41$, $p < .001$), with no difference between the Unit and Next Number tasks ($\beta = -0.02$, $p = .84$).

We next tested whether Resilient Counter's performance was affected by task by including an interaction between Counting Resilience and Task. A Likelihood Ratio Test indicated that the addition of this interaction term significantly improved the fit of the main effects model ($\chi^2_{(2)} = 19.52$, $p < .001$). In addition to main effects of Counting Resilience ($\chi^2_{(1)} = 15.61$, $p < .001$) and Task ($\chi^2_{(2)} = 518.38$, $p < .001$), this model also revealed a significant interaction between Resilience and Task ($\chi^2_{(2)} = 19.43$, $p < .001$). *Post hoc* contrasts indicated that the effect of Counting Resilience was equivalent on the Math Facts task in comparison to Unit and Next Number ($\beta = 0.17$, $p = .42$), being a Resilient Counter was associated with significantly more accurate performance on the Next Number task (which measures counting productivity) in comparison to the Unit Task ($\beta = 0.77$, $p < .001$).

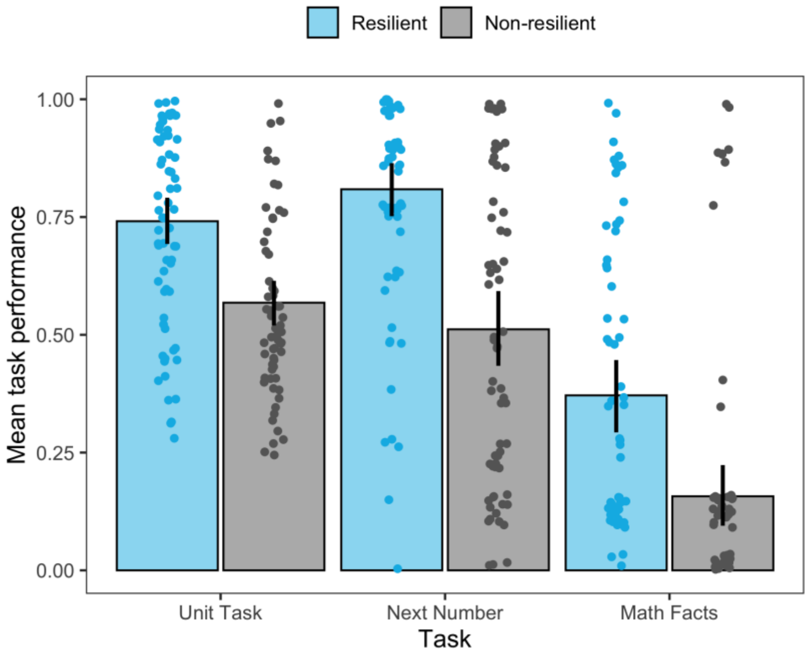

Figure 2. Mean task performance for Unit, Next Number, and Math Facts tasks, grouped by Counting Resilience. Each point represents an individual participants' mean performance and are jittered slightly to avoid overplotting. Error bars indicate 95% confidence intervals, computed by nonparametric bootstrap.