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SHORT-TIME REGULARITY FOR DYNAMIC EVOLUTION PROBLEMS IN PERFECT PLASTICITY

CLÉMENT MIFSUD

ABSTRACT. In this paper, we study the regularity properties of solutions of dynamic evolution problems in perfect plasticity. We prove that for any space dimension and for any closed convex set of constraints containing zero as an interior point, the solutions are regular in space during a short time interval if the data are smooth and compactly supported. The result is based on the hyperbolic structure of the model, namely the finite speed propagation property.

Keywords: Elasto-plasticity, regularity, finite speed propagation, comparison principle.

MSC (2010): 74C05, 35B65, 26A45, 35B51.

1. INTRODUCTION

Elasto-plastic models have been designed in order to mimic permanent deformations that can appear in a material submitted to a critical stress. From the experimental and theoretical points of view, one can observe discontinuity surfaces of the displacement.

To analyze such a phenomena, it has been necessary to introduce a new function space, namely the space of functions with bounded deformations, in [18, 20]. This problem was avoided in [13] thanks to a visco-plastic regularization. Unfortunately no information on the regularity of the strain tensor was proven. Thanks to the functional framework of [18, 20], the dynamical (in [1]) and the quasi-static case (in [21, 22]) have been studied and the existence and uniqueness problems have been solved. An important question remains in these two cases to know if one can expect more regularity for these solutions when the initial, boundary and source terms are smooth. This paper addresses the issue of regularity of the solution of the dynamical evolution of elasto-plastic materials.

To formulate the problem more precisely, we consider a reference configuration of a material, denoted by $\Omega \subset \mathbb{R}^n$, which is a bounded open set. We denote by $u : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ the displacement field. In the framework of small strain elasto-plasticity, the linearized strain tensor, $Eu := (Du + Du^T)/2$, is the sum of the elastic strain $e : [0, T] \times \Omega \rightarrow \mathbb{M}_{\text{sym}}^{n \times n}$ and the plastic strain

$$p : [0, T] \times \Omega \rightarrow \mathbb{M}_{\text{sym}}^{n \times n} \text{ i.e.}$$

$$Eu = e + p,$$

with $\mathbb{M}_{\text{sym}}^{n \times n}$ the space of real symmetric $n \times n$ matrices. The stress tensor $\sigma : [0, T] \times \Omega \rightarrow \mathbb{M}_{\text{sym}}^{n \times n}$ is defined, thanks to Hooke's law, as $\sigma := \mathbb{C}e$, where $\mathbb{C}$ is the symmetric fourth-order elasticity tensor. When the material is subjected to an external body force $f : [0, T] \times \Omega \rightarrow \mathbb{R}^n$, the equation of motion is

$$\ddot{u} - \text{div} \sigma = f \quad \text{in } [0, T] \times \Omega,$$

in the dynamical case or

$$-\text{div} \sigma = f \quad \text{in } [0, T] \times \Omega,$$

in the quasi-static case. In perfect plasticity, the stress tensor satisfies a constraint

$$\sigma \in K,$$

where K is a fixed closed convex subset of $\mathbb{M}_{\text{sym}}^{n \times n}$ and the plastic strain rate satisfies the flow rule

$$\dot{p} \in N_K(\sigma),$$

where $N_K(\sigma)$ is the normal cone to K at σ . Using convex analysis, this flow rule can be rewritten as Hill's principle of maximum plastic work

$$\sigma : \dot{p} = H(\dot{p}) := \max_{\tau \in K} (\tau : \dot{p}). \quad (1)$$

The function H is called the support function of the set K . Suitable boundary and initial conditions are supplemented to this set of equations.

In the dynamical case, one can find in [4] a theorem that states that the problem is well-posed for every K containing zero in its interior (in [1] the convex set of constraints is supposed to be bounded in the deviatoric direction). In this case, the only result available, to our knowledge, in the literature about the regularity of the solutions can be found in [3]. In that reference, the authors prove that the displacement field of a simplified (scalar) model of the dynamic elasto-plastic evolution, namely the anti-plane shear model, is H^1 regular in space (for small times) if the initial data and the source term are compactly supported and regular. The argument is based on the hyperbolic point of view of the evolution of the anti-plane shear model.

The purpose of our paper is to extend the regularity result of [3] to the general vectorial model of dynamical perfect plasticity. To be precise, our main theorem can be stated as follows: suppose that (u_0, v_0, e_0, p_0) is an initial data which is compactly supported with $\sigma_0 \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, that the function $f \in H^1(\Omega \times (0, T); \mathbb{R}^n)$ is such that for all $t \in [0, T]$, $f(t)$ is also compactly supported in space and that the (Dirichlet) boundary condition is homogeneous. Then, there exists a critical time T^* such that we have, for every time $\tau < T^*$, $u \in W^{1, \infty}([0, \tau]; H^1(\Omega; \mathbb{R}^n))$, $\sigma \in L^\infty(0, \tau; H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$ and $p \in W^{1, \infty}([0, \tau]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$.

To do so we first prove two propositions related to the hyperbolic point of view (see [3]) that allows us to say that information propagate at a finite speed. In fact, in the elasticity framework (*i.e.* without constraint), the existence and uniqueness of solutions to the classical (linear) elasto-dynamics have already been obtained thanks to hyperbolic tools (see [16]). For the elasto-plastic case and following [12, 19], one can prove (see [3, Section 3.3 & 7]) that the solutions of the dynamical elasto-plastic anti-plane shear model are also weak solutions of a related hyperbolic constrained problems (and a partial reciprocal statement holds true).

In the quasistatic case, there exist several works on the regularity of the stress tensor. In [5, 11], the authors focus on convex set of constraints K bounded in the deviatoric direction and are able to prove, with different methods, that $\sigma(t)$ belongs to H^1_{loc} for almost every $t \in [0, T]$. For the Norton/Hoff model, one can also find results about regularity up to the boundary in [14, 17]. In this work, we will not deal with quasistatic models. In fact, our approach cannot be extended to quasistatic solutions. Indeed, although the solutions of the quasistatic evolution could be obtained as limits of dynamic processes, rescaled in time thanks to a small parameter ϵ , the time T^* , during which one can obtain regular solutions for the dynamical evolution if the data are smooth and compactly supported, would degenerate as ϵ goes to 0. Consequently, at the limit $\epsilon \rightarrow 0$, one can not hope to control the space derivatives, on a small time interval, for the solutions of the quasistatic evolution.

The paper is organized as follows: in Section 2, we recall some useful notations and results about the dynamical elasto-plastic problem. Section 3 is devoted to prove our main theorem. The proof of our main theorem follows the lines of the regularity result of [3]. Although the comparison principle and its proof are very similar to Kato's inequality in [3], the rest of our arguments needs to be adapted to the general case of perfect plasticity. Indeed, due to the fact that we are dealing with any closed and convex elasticity sets K , the dissipation functional H is now a general positively 1-homogeneous and convex function of the plastic strain measure (in contrast with [3] where the specific choice of the elasticity set leads to a dissipation functional which is just the total variation of the plastic strain measure). Thus, a particular care and measure theoretic arguments (due to the use of convex functions of a measure) are needed to prove our short-time regularity result. Finally, a better choice of test functions in the comparison principle allows us to slightly improve the time regularity of the solution.

2. THE MATHEMATICAL SETTING

2.1. General notation. If a and $b \in \mathbb{R}^n$, we write $a \cdot b$ for the Euclidean scalar product, and we denote by $|a| = \sqrt{a \cdot a}$ the associated norm.

We write $\mathbb{M}^{n \times n}$ for the set of real $n \times n$ matrices, and $\mathbb{M}_{\text{sym}}^{n \times n}$ for that of all real symmetric $n \times n$ matrices. Given two matrices $A, B \in \mathbb{M}^{n \times n}$, we let $A : B := \text{tr}(AB^T)$ (B^T is the transpose of B) be the usual Euclidean scalar product and $|A| := \sqrt{\text{tr}(AA^T)}$ its associated norm over $\mathbb{M}^{n \times n}$. We recall that for any two vectors a and $b \in \mathbb{R}^n$, $a \otimes b \in \mathbb{M}^{n \times n}$ stands for the tensor product, i.e., $(a \otimes b)_{ij} = a_i b_j$ for all $1 \leq i, j \leq n$, and $a \odot b := (a \otimes b + b \otimes a)/2 \in \mathbb{M}_{\text{sym}}^{n \times n}$ denotes the symmetric tensor product. One has

$$\forall a, b \in \mathbb{R}^n, \quad \frac{1}{\sqrt{2}} |a| |b| \leq |a \odot b| \leq |a| |b|. \quad (2)$$

2.2. Function spaces. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. We use standard notations for Lebesgue and Sobolev spaces.

For E a locally compact subset of $\mathbb{R}^n$ and X an Euclidean space, we write $\mathcal{M}(E; X)$ (or simply $\mathcal{M}(E)$ if $X = \mathbb{R}$) for the space of bounded Radon measures on E with values in X , endowed with the norm $|\mu|(E)$, where $|\mu| \in \mathcal{M}(E)$ is the total variation of the measure μ .

The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, and the $(n-1)$ -dimensional Hausdorff measure by $\mathcal{H}^{n-1}$.

The Hilbert space, denoted $H(\text{div}, \Omega)$, of functions $\sigma \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ such that $\text{div} \sigma \in L^2(\Omega; \mathbb{R}^n)$ is endowed with the norm

$$\|\sigma\|_{H(\text{div}, \Omega)} = \left(\|\sigma\|_{L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})}^2 + \|\text{div} \sigma\|_{L^2(\Omega; \mathbb{R}^n)}^2 \right)^{\frac{1}{2}}.$$

The space $H(\text{div}, \Omega)$ obeys to the following trace theorem (see [7, Lemma 2.3], [23, Chapter 1, Section 1.3] and also [8, Chapter IX, Part A, Section 1.2, Theorem 1]):

Theorem 2.1. *For $\Omega \subset \mathbb{R}^n$ a bounded open set with Lipschitz boundary, we have*

- (1) *the space $\mathcal{C}^\infty(\overline{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})$ is dense in $H(\text{div}, \Omega)$;*
- (2) *the trace map, $\gamma_n : \sigma \mapsto \sigma \nu_{|\partial\Omega}$ defined on $\mathcal{C}^\infty(\overline{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})$, can be extended into a continuous linear mapping, still denoted γ_n , from $H(\text{div}, \Omega)$ to the space $H^{-\frac{1}{2}}(\partial\Omega; \mathbb{R}^n)$;*
- (3) *the kernel of γ_n is the closure of $\mathcal{C}_c^\infty(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ in $H(\text{div}, \Omega)$ and is denoted $H_0(\text{div}, \Omega)$.*

Remark 2.2. *Let Ω' be a bounded smooth open subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset \Omega'$. One should notice that if $\sigma \in H_0(\text{div}, \Omega)$, the extension of σ by the null-matrix on Ω' belongs to $H_0(\text{div}, \Omega')$.*

In addition, we denote, for any function ψ depending on the space variables (and possibly also on the time variable) and with values in $\mathbb{R}$, by $\nabla \psi$ its (spatial) distributional gradient.

Finally, $BD(\Omega)$ stands for the space of functions of bounded deformations in Ω . A function u belongs to $BD(\Omega)$ if $u \in L^1(\Omega; \mathbb{R}^n)$ and $Eu := (Du + Du^T)/2 \in \mathcal{M}(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, where Du is the distributional gradient of u (see [23] for a detailed presentation of this space).

2.3. The elastic energy. Let $\mathbb{C}$ be a fourth-order tensor satisfying

$$\forall (i, j, k, l) \in \{1, 2, \dots, n\}^4, \quad \mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jikl}. \quad (3)$$

We also assume that the operator $\mathbb{C}$ is coercive and continuous *i.e.* there exist two positive constant $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$ such that

$$\forall e \in \mathbb{M}_{\text{sym}}^{n \times n}, \quad \alpha_{\mathbb{C}}|e|^2 \leq \mathbb{C}e : e, \quad (4)$$

and

$$\forall e \in \mathbb{M}_{\text{sym}}^{n \times n}, \quad |\mathbb{C}e| \leq \beta_{\mathbb{C}}|e|. \quad (5)$$

To shorten notation, we set for $e \in \mathbb{M}_{\text{sym}}^{n \times n}$

$$Q(e) := \mathbb{C}e : e, \quad (6)$$

and for any $e \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$.

2.4. The set of constraints. Let K be a closed convex subset of $\mathbb{M}_{\text{sym}}^{n \times n}$ containing the origin as an interior point *i.e.* there exists a positive number α_H such that

$$\{\sigma \in \mathbb{M}_{\text{sym}}^{n \times n}, |\sigma| \leq \alpha_H\} \subset K. \quad (7)$$

In this work, we do not assume that K is bounded in the deviatoric direction.

2.5. Sets of admissible displacements and strains. For $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, we define the set of admissible displacements and strains, denoted $\mathcal{A}(w)$, as

$$\begin{aligned} \mathcal{A}(w) = \{ (u, e, p) \in (BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \times \mathcal{M}(\bar{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n}) \\ \text{s.t. } Eu = e + p \text{ in } \Omega \text{ and } p = (w - u) \odot \nu \mathcal{H}^{n-1} \text{ on } \partial\Omega \}. \end{aligned}$$

One can prove (following the lines of [7, Lemma 2.1]) that the multi-valued map $w \mapsto \mathcal{A}(w)$ enjoys the following closure property

Lemma 2.3. *Let (w_k) be a sequence in $H^1(\mathbb{R}^n; \mathbb{R}^n)$ and $(u_k, e_k, p_k) \in \mathcal{A}(w_k)$ such that there exists $(u_\infty, e_\infty, p_\infty) \in (BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \times \mathcal{M}(\bar{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})$ satisfying*

$$\begin{cases} u_k \xrightarrow{*} u_\infty & \text{in } BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n), \\ e_k \rightharpoonup e_\infty & \text{in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \\ p_k \xrightarrow{*} p_\infty & \text{in } \mathcal{M}(\bar{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n}), \end{cases}$$

and

$$w_k \rightharpoonup w_\infty \text{ in } H^1(\mathbb{R}^n; \mathbb{R}^n).$$

Then, $(u_\infty, e_\infty, p_\infty) \in \mathcal{A}(w_\infty)$.

Using difference quotients, one can prove a similar result for functions depending on the time variable.

Lemma 2.4. *Assume that the functions $t \mapsto w(t)$, $t \mapsto u(t)$, $t \mapsto e(t)$ and $t \mapsto p(t)$ are absolutely continuous from $[0, T]$ to respectively, $H^1(\mathbb{R}^n; \mathbb{R}^n)$, $(BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n))$, $L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ and $\mathcal{M}(\bar{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})$ and satisfy*

$$\forall t \in [0, T], \quad (u(t), e(t), p(t)) \in \mathcal{A}(w(t)).$$

Then, $(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in \mathcal{A}(\dot{w}(t))$ for a.e. $t \in [0, T]$.

2.6. Generalized stress/strain duality. In order to give a sense to Hill's principle of maximum plastic work (1), one has to define a suitable duality pairing between the stress tensor and the plastic strain rate tensor. Even though the stress tensor is a squared integrable function and the plastic strain rate tensor only a measure, one can give a (distributional) sense to this duality pairing. We briefly mention how one can do that (for more details, see [4, Section 2.4]).

Definition 2.5. *Let $\sigma \in H(\text{div}, \Omega)$ and $p \in \mathcal{M}(\bar{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})$ be such that there exist $u \in BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)$, $e \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ and $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$ satisfying $(u, e, p) \in \mathcal{A}(w)$. We define the distribution $[\sigma : p]$ on $\mathbb{R}^n$ as, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,*

$$\begin{aligned} \langle [\sigma : p], \varphi \rangle &= \int_{\Omega} \varphi(w - u) \cdot \text{div} \sigma \, dx \\ &\quad + \int_{\Omega} \sigma : ((w - u) \odot \nabla \varphi) \, dx + \int_{\Omega} \sigma : (Ew - e) \varphi \, dx. \end{aligned}$$

Remark 2.6. *One can check that Definition 2.5 is independent of the choice of (u, e, w) . In addition, since the distribution $[\sigma : p]$ is supported in $\bar{\Omega}$, one can define the duality product $\langle \sigma, p \rangle$ as*

$$\langle \sigma, p \rangle = \langle [\sigma : p], 1 \rangle = \int_{\Omega} (w - u) \cdot \text{div} \sigma \, dx + \int_{\Omega} \sigma : (Ew - e) \, dx.$$

2.7. The dissipated energy. The support function of K is a convex, lower semicontinuous and positively 1-homogeneous function and is denoted by $H : \mathbb{M}_{\text{sym}}^{n \times n} \rightarrow [0; +\infty]$, and one has

$$\forall q \in \mathbb{M}_{\text{sym}}^{n \times n}, \quad H(q) = \sup_{\tau \in K} \tau : q.$$

Using (7), we know that

$$\forall q \in \mathbb{M}_{\text{sym}}^{n \times n}, \quad \alpha_h |q| \leq H(q).$$

According to [15], for any $p \in \mathcal{M}(\bar{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})$, we define the nonnegative Borel measure on $\bar{\Omega}$ by

$$H(p)(B) = \int_B H \left(\frac{dp}{d|p|} \right) d|p|, \quad (8)$$

for any Borel set $B \subset \overline{\Omega}$. This measure is, in general, not locally finite. In the case when $H(p)$ has a finite mass, we set

$$\mathcal{H}(p) := H(p)(\overline{\Omega}),$$

and in this case, one can use the results about convex functions of measures of [9, 10] to derive the following duality formula

$$\mathcal{H}(p) = \sup_{\sigma \in \mathcal{C}^\infty(\overline{\Omega}; K)} \int_{\overline{\Omega}} \sigma : dp.$$

For any $p \in \mathcal{M}(\overline{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})$ such that $H(p)(\overline{\Omega}) < +\infty$ and that there exists $u \in BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)$, $e \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ and $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$ satisfying $(u, e, p) \in \mathcal{A}(w)$, we have for all $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ with $\varphi \geq 0$ (see [4, Equation (2.12)]),

$$\int_{\overline{\Omega}} \varphi dH(p) = \sup_{\substack{\sigma \in L^2(\Omega; K) \\ \text{with } \text{div} \sigma \in L^2(\Omega; \mathbb{R}^n)}} \langle [\sigma : p], \varphi \rangle. \quad (9)$$

Remark 2.7. *In particular, for every $\tau \in L^2(\Omega; K) \cap H(\text{div}, \Omega)$ and every $p \in \mathcal{M}(\overline{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})$ such that $H(p)(\overline{\Omega}) < +\infty$ and that there exists $u \in BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)$, $e \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ and $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$ satisfying $(u, e, p) \in \mathcal{A}(w)$, one has*

$$H(p) \geq [\tau : p] \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

2.8. Well-posedness of the dynamical elasto-plastic problem. We recall here one of the results of [4] that states that the dynamic elasto-plastic evolution problem is well-posed.

Theorem 2.8 (Theorem 5.1 in [4]). *Assume (3), (4), (5) and (7). Let $f \in AC([0, T]; L^2(\Omega; \mathbb{R}^n))$, $w \in H^2([0, T]; H^1(\mathbb{R}^n; \mathbb{R}^n)) \cap H^3([0, T]; L^2(\mathbb{R}^n; \mathbb{R}^n))$, $u_0 \in BD(\Omega) \cap L^2(\Omega; \mathbb{R}^n)$, $e_0 \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, $p_0 \in \mathcal{M}(\overline{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})$ and $v_0 \in H^1(\Omega; \mathbb{R}^n)$ satisfying: the additive decomposition*

$$Eu_0 = e_0 + p_0 \quad \text{in } \Omega,$$

the boundary condition

$$p_0 = (w(0) - u_0) \odot \nu \mathcal{H}^{n-1} \quad \text{on } \partial\Omega,$$

and the compatibility conditions

$$v_0 = \dot{w}(0) \quad \mathcal{H}^{n-1} - \text{a.e on } \partial\Omega \quad \text{and} \quad -\text{div} \sigma_0 = f(0), \quad \text{a.e in } \Omega,$$

with $\sigma_0 := \mathbb{C}e_0 \in K$. Then there exist a unique (u, e, σ, p) such that

$$\begin{cases} u \in AC([0, T]; BD(\Omega)) \cap W^{2, \infty}([0, T]; L^2(\Omega; \mathbb{R}^n)), \\ e, \sigma \in W^{1, \infty}([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})), \\ p \in AC([0, T]; \mathcal{M}(\overline{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})), \end{cases}$$

and satisfying, for every $t \in [0, T]$, the additive decomposition

$$Eu(t) = e(t) + p(t) \text{ in } \Omega,$$

the boundary condition

$$p(t) = (w(t) - u(t)) \odot \nu \mathcal{H}^{n-1} \quad \text{on } \partial\Omega,$$

the stress constraint

$$\sigma(t) \in K \quad \text{a.e in } \Omega \quad \text{with } \sigma(t) = \mathbb{C}e(t),$$

the equation of motion

$$\ddot{u} - \operatorname{div} \sigma = f, \quad \text{a.e in } (0, T) \times \Omega,$$

the initial condition

$$(u(0), \dot{u}(0), e(0), p(0)) = (u_0, v_0, e_0, p_0),$$

and for a.e. $t \in [0, T]$ the distribution $[\sigma(t) : \dot{p}(t)]$ is a measure in $\mathcal{M}(\overline{\Omega})$ satisfying the flow rule

$$H(\dot{p}(t)) = [\sigma(t) : \dot{p}(t)] \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

3. SHORT-TIME REGULARITY FOR DYNAMIC ELASTO-PLASTIC PROBLEMS

The goal of this section is to state and prove our main result, Theorem 3.1, about short-time regularity of elasto-plastic solutions in the dynamical case.

Theorem 3.1. *Under the same assumptions as in [4, Theorem 5.1], recalled in Section 2.8, if, in addition, the boundary source term $w = 0$, and if there exists a compact set $\mathcal{K} \subset \Omega$ such that*

$$\operatorname{supp}(u_0, v_0, \sigma_0, p_0, f(t)) \subset \mathcal{K} \quad \text{for all } t \in [0, T], \quad (10)$$

and that

$$\sigma_0 \in H^1(\Omega; \mathbb{M}_{\operatorname{sym}}^{n \times n}), \quad f \in H^1((0, T) \times \Omega; \mathbb{R}^n).$$

Then the solution (u, e, σ, p) of the elasto-plastic problem given in Section 2.8 for the initial condition $(u_0, v_0, \sigma_0, p_0)$ and the source terms w and f satisfies the following regularity properties: setting $T^* := \min\left(\frac{\sqrt{\alpha_C}}{\beta_C} \operatorname{dist}(\mathcal{K}, \partial\Omega), T\right)$, we have for all $\tau \in (0, T^*)$

$$\begin{cases} \dot{u} \in L^\infty(0, \tau; H^1(\Omega; \mathbb{R}^n)), \\ \sigma \in L^\infty(0, \tau; H^1(\Omega; \mathbb{M}_{\operatorname{sym}}^{n \times n})). \end{cases}$$

Moreover, if $p_0 \in L^2(\Omega; \mathbb{M}_{\operatorname{sym}}^{n \times n})$ and $u_0 \in H^1(\Omega; \mathbb{R}^n)$, we obtain

$$\begin{cases} u \in W^{1, \infty}([0, \tau]; H^1(\Omega; \mathbb{R}^n)), \\ p \in W^{1, \infty}([0, \tau]; L^2(\Omega; \mathbb{M}_{\operatorname{sym}}^{n \times n})). \end{cases}$$

To prove Theorem 3.1, we establish two intermediate results. The first one, Proposition 3.2, is a reminiscence of Kato's inequality of [12] (and its proof closely follows the lines of [3, Proposition 5.12]) and allows us to get a comparison principle between two solutions of the elasto-plastic problem (see Remark 3.3). This result is obtained under the hypothesis that the boundary source term does not depend on the time variable. This hypothesis will be assumed in the whole section.

The second one, Proposition 3.4, shows that if the initial velocity v_0 and the initial stress σ_0 are compactly supported in space, and also if the source term $f(t)$ is compactly supported in space (for all $t \in [0, T]$) then there exists a time T^* such that the velocity and the stress will remain compactly supported in space in $[0, T^*]$. As explained in Section 1, this part needs non trivial technical adaptations of the result of [3] because of the presence of the general support function H of K (instead of simply the total variation).

Using these two results and since all functions involved here are null in a neighborhood of the boundary, one can apply the so-called method of translations (see [6]) to get the short-time regularity result. This regularity result is, as we previously said in Section 1, slightly improved with respect to the time regularity compared to the regularity result of [3].

3.1. Comparison principle. In the sequel, we suppose that the boundary terms do not depend on the time variable.

Proposition 3.2. *Let (u, e, σ, p) and $(\tilde{u}, \tilde{e}, \tilde{\sigma}, \tilde{p})$ be two solutions of the elasto-plastic problem associated with the initial data (u_0, v_0, e_0, p_0) (respectively $(\tilde{u}_0, \tilde{v}_0, \tilde{e}_0, \tilde{p}_0)$) and the source terms w and f (respectively $\tilde{w}$ and $\tilde{f}$). Then for all $\varphi \in W^{1,\infty}((0, T) \times \Omega)$ with $\varphi \geq 0$,*

$$\begin{aligned} & \int_0^T \int_{\Omega} |\dot{u} - \dot{\tilde{u}}|^2 \varphi \, dx \, dt + \int_0^T \int_{\Omega} Q(e - \tilde{e}) \dot{\varphi} \, dx \, dt \\ & - 2 \int_0^T \int_{\Omega} (\sigma - \tilde{\sigma}) : ((\dot{u} - \dot{\tilde{u}}) \odot \nabla \varphi) \, dx \, dt + 2 \int_0^T \int_{\Omega} (f - \tilde{f}) \cdot (\dot{u} - \dot{\tilde{u}}) \varphi \, dx \, dt \\ & + \int_{\Omega} |v_0 - \tilde{v}_0|^2 \varphi(0) \, dx + \int_{\Omega} Q(e_0 - \tilde{e}_0) \varphi(0) \, dx \geq 0. \end{aligned} \quad (11)$$

Proof. Let (u, e, σ, p) (resp. $(\tilde{u}, \tilde{e}, \tilde{\sigma}, \tilde{p})$) be a solution of the elasto-plastic problem associated with the initial condition (u_0, v_0, e_0, p_0) (resp. $(\tilde{u}_0, \tilde{v}_0, \tilde{e}_0, \tilde{p}_0)$) and the source terms f and w (resp. $\tilde{f}$ and $\tilde{w}$). Thanks to the equations of motion, we know that

$$\ddot{u} - \ddot{\tilde{u}} - (\operatorname{div} \sigma - \operatorname{div} \tilde{\sigma}) = f - \tilde{f}, \quad \text{a.e in } (0, T) \times \Omega.$$

Taking the scalar product with the function $(\dot{u} - \dot{\tilde{u}})\varphi \in L^2(0, T; L^2(\Omega; \mathbb{R}^n))$ for $\varphi \in W^{1,\infty}((0, T) \times \Omega)$ with $\varphi \geq 0$, we infer that

$$\begin{aligned} \int_0^T \int_{\Omega} (\ddot{u} - \ddot{\tilde{u}}) \cdot (\dot{u} - \dot{\tilde{u}}) \varphi \, dx \, dt - \int_0^T \int_{\Omega} (\operatorname{div} \sigma - \operatorname{div} \tilde{\sigma}) \cdot (\dot{u} - \dot{\tilde{u}}) \varphi \, dx \, dt \\ = \int_0^T \int_{\Omega} (f - \tilde{f}) \cdot (\dot{u} - \dot{\tilde{u}}) \varphi \, dx \, dt. \end{aligned} \quad (12)$$

Using Definition 2.5 along with Lemma 2.4 and since the functions w and $\tilde{w}$ do not depend on the time variable, we get that, for a.e. $t \in [0, T]$,

$$\begin{aligned} \langle [\sigma(t) : (\dot{p}(t) - \dot{\tilde{p}}(t))], \varphi(t) \rangle &= - \int_{\Omega} \varphi(t) (\dot{u}(t) - \dot{\tilde{u}}(t)) \cdot \operatorname{div} \sigma(t) \, dx \\ &- \int_{\Omega} \sigma(t) : ((\dot{u}(t) - \dot{\tilde{u}}(t)) \odot \nabla \varphi(t)) \, dx - \int_{\Omega} \sigma(t) : (\dot{e}(t) - \dot{\tilde{e}}(t)) \varphi(t) \, dx, \end{aligned} \quad (13)$$

and similarly,

$$\begin{aligned} \langle [\tilde{\sigma}(t) : (\dot{p}(t) - \dot{\tilde{p}}(t))], \varphi(t) \rangle &= - \int_{\Omega} \varphi(t) (\dot{u}(t) - \dot{\tilde{u}}(t)) \cdot \operatorname{div} \tilde{\sigma}(t) \, dx \\ &- \int_{\Omega} \tilde{\sigma}(t) : ((\dot{u}(t) - \dot{\tilde{u}}(t)) \odot \nabla \varphi(t)) \, dx - \int_{\Omega} \tilde{\sigma}(t) : (\dot{e}(t) - \dot{\tilde{e}}(t)) \varphi(t) \, dx. \end{aligned} \quad (14)$$

Subtracting (13) and (14), we have

$$\begin{aligned} \langle [(\sigma(t) - \tilde{\sigma}(t)) : (\dot{p}(t) - \dot{\tilde{p}}(t))], \varphi(t) \rangle \\ = - \int_{\Omega} (\sigma(t) - \tilde{\sigma}(t)) : (\dot{e}(t) - \dot{\tilde{e}}(t)) \varphi(t) \, dx \\ - \int_{\Omega} \varphi(t) (\dot{u}(t) - \dot{\tilde{u}}(t)) \cdot (\operatorname{div} \sigma(t) - \operatorname{div} \tilde{\sigma}(t)) \, dx \\ - \int_{\Omega} (\sigma(t) - \tilde{\sigma}(t)) : ((\dot{u}(t) - \dot{\tilde{u}}(t)) \odot \nabla \varphi(t)) \, dx. \end{aligned} \quad (15)$$

We integrate in time the previous equality and use the relation between σ and e (respectively $\tilde{\sigma}$ and $\tilde{e}$) to get

$$\begin{aligned} - \int_0^T \int_{\Omega} (\operatorname{div} \sigma - \operatorname{div} \tilde{\sigma}) \cdot (\dot{u} - \dot{\tilde{u}}) \varphi \, dx \, dt \\ = \int_0^T \int_{\Omega} (\sigma - \tilde{\sigma}) : ((\dot{u} - \dot{\tilde{u}}) \odot \nabla \varphi) \, dx \, dt \\ + \int_0^T \int_{\Omega} (e - \tilde{e}) : \mathbb{C} (\dot{e} - \dot{\tilde{e}}) \varphi \, dx \, dt + \int_0^T \langle [(\sigma - \tilde{\sigma}) : (\dot{p} - \dot{\tilde{p}})], \varphi \rangle \, dt. \end{aligned} \quad (16)$$

On the one hand, by the flow rule we have for a.e. $t \in [0, T]$,

$$[\sigma(t) : \dot{p}(t)] = H(\dot{p}(t)) \quad \text{and} \quad [\tilde{\sigma}(t) : \dot{\tilde{p}}(t)] = H(\dot{\tilde{p}}(t)),$$

as measures on $\overline{\Omega}$. On the other hand, the stress constraint and Remark 2.7 imply that for a.e. $t \in [0, T]$

$$[\tilde{\sigma}(t) : \dot{p}(t)] \leq H(\dot{p}(t)) \quad \text{and} \quad [\sigma(t) : \dot{\tilde{p}}(t)] \leq H(\dot{\tilde{p}}(t)) \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

Consequently, since φ is non-negative, the integral term of (16)

$$\int_0^T \langle [(\sigma - \tilde{\sigma}) : (\dot{p} - \dot{\tilde{p}})], \varphi \rangle dt,$$

is non-negative. It implies that

$$\begin{aligned} 0 \leq & - \int_0^T \int_{\Omega} (\ddot{u} - \ddot{\tilde{u}}) \cdot (\dot{u} - \dot{\tilde{u}}) \varphi dx dt - \int_0^T \int_{\Omega} (e - \tilde{e}) : \mathbb{C} (\dot{e} - \dot{\tilde{e}}) \varphi dx dt \\ & - \int_0^T \int_{\Omega} (\sigma - \tilde{\sigma}) : ((\dot{u} - \dot{\tilde{u}}) \odot \nabla \varphi) dx dt + \int_0^T \int_{\Omega} (f - \tilde{f}) \cdot (\dot{u} - \dot{\tilde{u}}) \varphi dx dt. \end{aligned} \quad (17)$$

Since $\dot{u} - \dot{\tilde{u}} \in W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{R}^n))$ and $\sigma - \tilde{\sigma} \in W^{1,\infty}([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$, we can integrate by parts with respect to the time variable to get that

$$\begin{aligned} & - \int_0^T \int_{\Omega} (\ddot{u} - \ddot{\tilde{u}}) \cdot (\dot{u} - \dot{\tilde{u}}) \varphi dx dt - \int_0^T \int_{\Omega} (e - \tilde{e}) : \mathbb{C} (\dot{e} - \dot{\tilde{e}}) \varphi dx dt \\ & = \frac{1}{2} \left(\int_0^T \int_{\Omega} |\dot{u} - \dot{\tilde{u}}|^2 \dot{\varphi} dx dt + \int_0^T \int_{\Omega} Q(e - \tilde{e}) \dot{\varphi} dx dt \right. \\ & \quad \left. + \int_{\Omega} |v_0 - \tilde{v}_0|^2 \varphi(0) dx + \int_{\Omega} Q(e_0 - \tilde{e}_0) \varphi(0) dx \right. \\ & \quad \left. - \int_{\Omega} |\dot{u}(T) - \dot{\tilde{u}}(T)|^2 \varphi(T) dx - \int_{\Omega} Q(e(T) - \tilde{e}(T)) \varphi(T) dx \right), \end{aligned} \quad (18)$$

and substituting (18) into (17) leads to

$$\begin{aligned} 0 \leq & \frac{1}{2} \left(\int_0^T \int_{\Omega} |\dot{u} - \dot{\tilde{u}}|^2 \dot{\varphi} dx dt + \int_0^T \int_{\Omega} Q(e - \tilde{e}) \dot{\varphi} dx dt \right. \\ & \left. + \int_{\Omega} |v_0 - \tilde{v}_0|^2 \varphi(0) dx + \int_{\Omega} Q(e_0 - \tilde{e}_0) \varphi(0) dx \right) \\ & - \int_0^T \int_{\Omega} (\sigma - \tilde{\sigma}) : ((\dot{u} - \dot{\tilde{u}}) \odot \nabla \varphi) dx dt + \int_0^T \int_{\Omega} (f - \tilde{f}) \cdot (\dot{u} - \dot{\tilde{u}}) \varphi dx dt, \end{aligned}$$

since $\varphi(T) \geq 0$. $\square$

Remark 3.3. Thanks to Proposition 3.2 with the test function $\varphi(t, x) = \frac{T-t}{T}$, we get that

$$\begin{aligned} & \int_0^T \int_{\Omega} |\dot{u} - \dot{\tilde{u}}|^2 \, dx \, dt + \int_0^T \int_{\Omega} Q(e - \tilde{e}) \, dx \, dt \\ & \leq T \int_{\Omega} |v_0 - \tilde{v}_0|^2 \, dx + T \int_{\Omega} Q(e_0 - \tilde{e}_0) \, dx + 2T \int_0^T \int_{\Omega} (f - \tilde{f}) \cdot (\dot{u} - \dot{\tilde{u}}) \varphi \, dx \, dt, \end{aligned}$$

from which we deduce that there exists a constant C depending only on T such that the following comparison principle holds

$$\begin{aligned} & \int_0^T \int_{\Omega} |\dot{u} - \dot{\tilde{u}}|^2 \, dx \, dt + \int_0^T \int_{\Omega} Q(e - \tilde{e}) \, dx \, dt \\ & \leq C \left(\int_{\Omega} |v_0 - \tilde{v}_0|^2 \, dx + \int_{\Omega} Q(e_0 - \tilde{e}_0) \, dx + \int_0^T \int_{\Omega} |f - \tilde{f}|^2 \, dx \, dt \right). \end{aligned}$$

3.2. Finite speed propagation property.

Proposition 3.4. Let (u, e, σ, p) be the solution of the elasto-plastic problem given in Section 2.8 for the initial condition $(u_0, v_0, \sigma_0, p_0)$ and the source terms w and f . Suppose that there exists a compact set $\mathcal{K} \subset \Omega$ such that

$$\text{supp}(v_0, \sigma_0, f(t)) \subset \mathcal{K} \quad \text{for all } t \in [0, T].$$

Then, for all $T^* \in (0, T]$ be such that $T^* < \frac{\sqrt{\alpha_C}}{\beta_C} \text{dist}(\mathcal{K}, \partial\Omega)$, there exists a compact set $\mathcal{K}^* \subset \Omega$ such that $\text{supp}(\dot{u}, \sigma) \subset [0, T^*] \times \mathcal{K}^*$.

Remark 3.5. Since $\mathbb{C}$ is invertible, the elastic tensor e , which is such that $\sigma = \mathbb{C}e$, is also compactly supported due to Proposition 3.4 and $\text{supp}(e) \subset [0, T^*] \times \mathcal{K}^*$.

Moreover, if one assumes that $\text{supp}(u_0) \subset \mathcal{K}^*$, one obtain, thanks to Proposition 3.4, that $\text{supp}(u) \subset [0, T^*] \times \mathcal{K}^*$. Indeed, we have, as a Bochner integral in $L^2(\Omega; \mathbb{R}^n)$, for all $t \in [0, T]$,

$$u(t) = u_0 + \int_0^t \dot{u}(s) \, ds,$$

since $u \in W^{2,\infty}([0, T]; L^2(\Omega; \mathbb{R}^n))$. Consequently, if we assume that the support of u_0 is a subset of $\mathcal{K}^*$, the measure Eu is also compactly supported in $[0, T^*] \times \mathcal{K}^*$ and we deduce, thanks to the additive decomposition of Eu , that $\text{supp}(p) \subset [0, T^*] \times \mathcal{K}^*$.

Proof. By assumption, we know that for all $x \in \partial\Omega$, we have $\frac{\sqrt{\alpha_C}}{\beta_C} \text{dist}(x, \mathcal{K}) > T^*$, so that we can find some $r_x > 0$ such that $\text{dist}(x, \mathcal{K}) = r_x + \frac{T^* \beta_C}{\sqrt{\alpha_C}}$.

Using the fact that $\partial\Omega$ is compact, we obtain the existence of $p \in \mathbb{N}$ and $x_1, \dots, x_p \in \partial\Omega$ such that

$$\partial\Omega \subset \bigcup_{i=1}^p B\left(x_i, \frac{r_{x_i}}{4}\right),$$

and we define $\eta = \min_{1 \leq i \leq p} r_{x_i}/4 > 0$, and the boundary layer

$$L_\eta = \{y \in \Omega : 0 < \text{dist}(y, \partial\Omega) < \eta\} \subset \bigcup_{i=1}^p B\left(x_i, \frac{r_{x_i}}{2}\right) \cap \Omega.$$

First, we prove that $v_0 = 0$, $\sigma_0 = 0$ and $f(t) = 0$ in the boundary layer L_η for all $t \in [0, T]$. Indeed, for every $y \in B\left(x_i, \frac{r_{x_i}}{2}\right)$ where $1 \leq i \leq p$, we have

$$|\text{dist}(y, \mathcal{K}) - \text{dist}(x_i, \mathcal{K})| \leq |y - x_i| \leq \frac{r_{x_i}}{2},$$

and consequently,

$$\text{dist}(y, \mathcal{K}) \geq \frac{T^* \beta_C}{\sqrt{\alpha_C}} + \frac{r_{x_i}}{2} > 0, \quad (19)$$

which implies that $v_0 = 0$, $\sigma_0 = 0$ and $f(t) = 0$ in $L_\eta \subset \bigcup_{i=1}^p B(x_i, r_i/2)$ for all $t \in [0, T]$ by definition of $\mathcal{K}$.

Let us show that $\dot{u} = 0$ and $\sigma = 0$ on $(0, T^*) \times L_\eta$. To this aim, we choose $x_0 \in L_\eta$, and, since the set L_η is open, one can find $\rho_0 \in (0, \eta/2)$ such that $B(x_0, \rho_0) \subset L_\eta$. Then we define the function $\varphi \in W^{1,\infty}((0, T^*) \times \mathbb{R}^n)$ as

$$\varphi(t, x) = \begin{cases} T^* - t + \frac{\sqrt{\alpha_C}}{\beta_C} (\rho_0 - |x - x_0|) & \text{if } \begin{cases} t \in [0, T^*], \\ \rho_0 < |x - x_0| < \rho_0 + \beta_C \frac{T^* - t}{\sqrt{\alpha_C}}, \end{cases} \\ T^* - t & \text{if } \begin{cases} t \in [0, T^*], \\ |x - x_0| < \rho_0, \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

The function φ is non-negative and its support is contained in $(0, T^*) \times B(x_0, \rho_0 + \frac{T^* \beta_C}{\sqrt{\alpha_C}})$. Consequently, using Proposition 3.2 (with (u, e, σ, p) and the null solution) it follows that

$$\begin{aligned} 2 \int_0^{T^*} \int_\Omega f \cdot \dot{u} \varphi \, dx \, dt + \int_\Omega |v_0|^2 \varphi(0) \, dx + \int_\Omega Q(e_0) \varphi(0) \, dx \geq \\ \int_0^{T^*} \int_\Omega -|\dot{u}|^2 \dot{\varphi} - Q(e) \dot{\varphi} + 2\sigma : (\dot{u} \odot \nabla \varphi) \, dx \, dt. \end{aligned} \quad (20)$$

Now, we are going to show that the integral part of the right-hand side of the inequality is non-negative *i.e.* that we have almost everywhere in $(0, T^*) \times \Omega$

$$-|\dot{u}|^2 \dot{\varphi} - Q(e) \dot{\varphi} + 2\sigma : (\dot{u} \odot \nabla \varphi) \geq 0. \quad (21)$$

Differentiating the function φ , it yields

$$\begin{cases} \dot{\varphi} = -\mathbf{1}_{\{(t,x): t \in [0, T^*], |x-x_0| < \rho_0 + \beta_C \frac{T^*-t}{\sqrt{\alpha_C}}\}}, \\ \nabla \varphi = -\frac{\sqrt{\alpha_C}}{\beta_C} \frac{x-x_0}{|x-x_0|} \mathbf{1}_{\{(t,x): t \in [0, T^*], \rho_0 < |x-x_0| < \rho_0 + \beta_C \frac{T^*-t}{\sqrt{\alpha_C}}\}}. \end{cases}$$

Consequently, on $\{(t, x) : t \in [0, T^*], |x - x_0| \leq \rho_0\}$, we have

$$-|\dot{u}|^2 \dot{\varphi} - Q(e) \dot{\varphi} + 2\sigma : (\dot{u} \odot \nabla \varphi) = |\dot{u}|^2 + Q(e) \geq 0.$$

While on $\{(x, t) : t \in [0, T^*], \rho_0 < |x - x_0| < \rho_0 + \beta_C \frac{T^*-t}{\sqrt{\alpha_C}}\}$, we have

$$-|\dot{u}|^2 \dot{\varphi} - Q(e) \dot{\varphi} + 2\sigma : (\dot{u} \odot \nabla \varphi) = |\dot{u}|^2 + Q(e) - 2 \frac{\sqrt{\alpha_C}}{\beta_C} \sigma : \left(\dot{u} \odot \left(\frac{x - x_0}{|x - x_0|} \right) \right).$$

Using (2), we obtain

$$-2 \frac{\sqrt{\alpha_C}}{\beta_C} \sigma : \left(\dot{u} \odot \left(\frac{x - x_0}{|x - x_0|} \right) \right) \geq -2 \frac{\sqrt{\alpha_C}}{\beta_C} |\sigma| |\dot{u}|.$$

Since $\sigma = \mathbb{C}e$, it yields on $\{(x, t) : t \in [0, T^*], \rho_0 < |x - x_0| < \rho_0 + \beta_C \frac{T^*-t}{\sqrt{\alpha_C}}\}$

$$-|\dot{u}|^2 \dot{\varphi} - Q(e) \dot{\varphi} + 2\sigma : (\dot{u} \odot \nabla \varphi) \geq |\dot{u}|^2 + Q(e) - 2 \frac{\sqrt{\alpha_C}}{\beta_C} |\mathbb{C}e| |\dot{u}|.$$

Property (5) implies

$$|\dot{u}|^2 + Q(e) - 2 \frac{\sqrt{\alpha_C}}{\beta_C} |\mathbb{C}e| |\dot{u}| \geq |\dot{u}|^2 + Q(e) - 2\sqrt{\alpha_C} |e| |\dot{u}|,$$

and by the coercivity property (4), Inequality (21) is satisfied a.e in $(0, T^*) \times \Omega$. Consequently, Inequalities (20) and (21) imply

$$\begin{aligned} \int_{\Omega} -|\dot{u}|^2 \dot{\varphi} - Q(e) \dot{\varphi} + 2\sigma : (\dot{u} \odot \nabla \varphi) \, dx \\ \geq \int_{B(x_0, \rho_0)} -|\dot{u}|^2 \dot{\varphi} - Q(e) \dot{\varphi} + 2\sigma : (\dot{u} \odot \nabla \varphi) \, dx. \end{aligned} \quad (22)$$

Using the definition of φ , (20) and (22), we obtain

$$\begin{aligned} - \int_0^{T^*} \int_{B(x_0, \rho_0)} (|\dot{u}|^2 + Q(e)) \dot{\varphi} \, dx \, dt + 2 \int_0^{T^*} \int_{B(x_0, \rho_0)} \sigma : (\dot{u} \odot \nabla \varphi) \, dx \, dt \\ \leq \int_{\mathcal{S}} (|v_0|^2 + Q(e_0)) \varphi(0) \, dx + 2 \int_0^{T^*} \int_{\mathcal{S}} |f| |\dot{u}| \varphi \, dx \, dt, \end{aligned}$$

where $\mathcal{S} \subset \Omega$ is such that $\text{supp}(\varphi) \subset [0, T^*] \times \mathcal{S}$. Since the spatial derivative of φ vanishes on $B(x_0, \rho_0)$, we get that

$$\begin{aligned} \int_0^{T^*} \int_{B(x_0, \rho_0)} (|\dot{u}|^2 + Q(e)) \, dx \, dt &\leq \int_{\mathcal{S}} (|v_0|^2 + Q(e_0)) \varphi(0) \, dx \\ &\quad + 2 \int_0^{T^*} \int_{\mathcal{S}} |f| |\dot{u}| \varphi \, dx \, dt. \end{aligned} \quad (23)$$

We now show that the right-hand side of (23) is null due to the definitions of x_0 , ρ_0 and L_η . Indeed, we obtain that for every $y \in \mathcal{S} \subset B\left(x_0, \rho_0 + \frac{T^* \beta_{\mathbb{C}}}{\sqrt{\alpha_{\mathbb{C}}}}\right)$ (by definition of φ),

$$|\text{dist}(y, \mathcal{K}) - \text{dist}(x_0, \mathcal{K})| \leq \rho_0 + \frac{T^* \beta_{\mathbb{C}}}{\sqrt{\alpha_{\mathbb{C}}}},$$

and thanks to (19) and by definition of η , we have for any $x_0 \in L_\eta$

$$\text{dist}(x_0, \mathcal{K}) > \frac{T^* \beta_{\mathbb{C}}}{\sqrt{\alpha_{\mathbb{C}}}} + \eta,$$

thus,

$$\text{dist}(y, \mathcal{K}) > \frac{T^* \beta_{\mathbb{C}}}{\sqrt{\alpha_{\mathbb{C}}}} + \eta - \rho_0 - \frac{T^* \beta_{\mathbb{C}}}{\sqrt{\alpha_{\mathbb{C}}}} > \frac{\eta}{2} > 0.$$

Consequently, the right-hand side of (23) vanishes and

$$\int_0^{T^*} \int_{B(x_0, \rho_0)} (|\dot{u}|^2 + Q(e)) \, dx \, dt \leq 0,$$

which implies that both $\dot{u}$ and e vanish in $(0, T^*) \times L_\eta$. Since $\sigma = \mathbb{C}e$, the conclusion thus follows by setting $\mathcal{K}^* = \Omega \setminus L_\eta$. $\square$

3.3. Short time regularity result. In the sequel, we suppose that the boundary source term w is identically equal to zero on $(0, T) \times \mathbb{R}^n$.

Proof of Theorem 3.1. Let $T^* = \min\left(\frac{\sqrt{\alpha_{\mathbb{C}}}}{\beta_{\mathbb{C}}} \text{dist}(\mathcal{K}, \partial\Omega), T\right)$ and $\tau < T^*$. We consider (u, e, σ, p) the solution on $[0, \tau] \times \Omega$ of the elasto-plastic problem given in Section 2.8 for the initial condition $(u_0, v_0, \sigma_0, p_0)$, the source term f and the null boundary source term. Thanks to Proposition 3.4 and also to Remark 3.5, we know that there exists a compact set, $\mathcal{K}^* \subset \Omega$ such that

$$\text{supp}(u, e, \sigma, p) \subset [0, \tau] \times \mathcal{K}^*. \quad (24)$$

Since $\mathcal{K}^*$ is a compact subset of Ω , there exists $\delta > 0$ such that for all $h \in \mathbb{R}^n$ with $|h| < \delta$, the sets $\mathcal{K}^* + h$ are also compactly embedded in Ω . Let Ω' be a bounded smooth open subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset \Omega'$, and for all $h \in \mathbb{R}^n$ with $|h| < \delta$, $\Omega + h \subset \Omega'$.

Step 1: Extension on $(0, \tau) \times \Omega'$. Since the boundary source term is null on $(0, T) \times \mathbb{R}^n$, we extend (f, u, e, p) by zero on $(0, \tau) \times \Omega'$ and we denote by $(\bar{f}, \bar{u}, \bar{e}, \bar{p})$ these extensions. We also extend by zero the initial data u_0, v_0, e_0, σ_0 and p_0 and denote by $\bar{u}_0, \bar{v}_0, \bar{e}_0, \bar{\sigma}_0$ and $\bar{p}_0$ these extensions.

Firstly, we check that these extensions are as regular as the solution on $(0, \tau) \times \Omega$.

The displacement field $\bar{u}$ belongs to $W^{2,\infty}([0, \tau]; L^2(\Omega'; \mathbb{R}^n))$, the elastic strain $\bar{e}$ to $W^{1,\infty}([0, \tau]; L^2(\Omega'; \mathbb{M}_{\text{sym}}^{n \times n}))$ and the plastic strain $\bar{p}$ to the space $AC([0, \tau]; \mathcal{M}(\overline{\Omega'}; \mathbb{M}_{\text{sym}}^{n \times n}))$. The source term $\bar{f}$ belongs to $H^1([0, \tau] \times \Omega'; \mathbb{R}^n)$. We also define the extension of the stress $\bar{\sigma} := \mathbb{C}\bar{e} \in W^{1,\infty}([0, \tau]; L^2(\Omega'; K))$.

In addition, for all $t \in [0, \tau]$, since the (inner) trace on $\partial\Omega$ of $u(t)$ vanishes and since we have extended u by zero, [23, Chapter 2, Proposition 2.1] ensures that the function $\bar{u}(t) \in BD(\Omega')$ and

$$\begin{aligned} E\bar{u}(t) &= Eu(t) && \text{on } \Omega, \\ E\bar{u}(t) &= 0 && \text{on } \Omega' \setminus \Omega. \end{aligned}$$

Hence, we get that $\bar{u} \in AC([0, \tau]; BD(\Omega'))$.

Now we show that these extensions (and their time derivatives) satisfy the additive decomposition and the boundary condition in Ω' . Indeed, we know that for every $t \in [0, \tau]$, since we extend u, e and p by zero outside Ω ,

$$E\bar{u}(t) = \bar{e}(t) + \bar{p}(t) \quad \text{in } \Omega', \quad (25)$$

and

$$p(t) = 0 = -u(t) \odot \nu \mathcal{H}^{n-1} \quad \text{on } \partial\Omega'. \quad (26)$$

These regularity properties together with (25) and (26) ensure, thanks to Lemma 2.4, that for a.e. $t \in [0, \tau]$ we have,

$$\dot{\bar{p}}(t) = E\dot{\bar{u}}(t) - \dot{\bar{e}}(t) \quad \text{on } \Omega',$$

and

$$\dot{\bar{p}}(t) = 0 \odot \nu \mathcal{H}^{n-1} \quad \text{on } \partial\Omega'.$$

Remark also that, since the support of σ is contained in $[0, \tau] \times \mathcal{K}$, we deduce that $\sigma(t)$ belongs to $H_0(\text{div}, \Omega)$ for all $t \in [0, \tau]$ and consequently (see Remark 2.2) $\bar{\sigma}(t) \in H_0(\text{div}, \Omega')$ for all $t \in [0, \tau]$. Consequently, it allows us to define for a.e. $t \in [0, \tau]$ the distribution $[\bar{\sigma}(t) : \dot{\bar{p}}(t)]$ as in Definition 2.5. Clearly

$$[\bar{\sigma}(t) : \dot{\bar{p}}(t)] = [\sigma(t) : \dot{p}(t)] \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \quad (27)$$

Then we are going to prove that for a.e. $t \in [0, \tau]$, $H(\dot{\bar{p}}(t))$ is in $\mathcal{M}(\overline{\Omega'})$ and that we have

$$H(\dot{\bar{p}}(t)) = [\bar{\sigma}(t) : \dot{\bar{p}}(t)] \quad \text{in } \mathcal{M}(\overline{\Omega'}).$$

First, observe that we only need to prove that

$$H(\dot{\bar{p}}(t))(\overline{\Omega'}) < +\infty,$$

for a.e. $t \in [0, \tau]$ to get that $H(\dot{p}(t)) \in \mathcal{M}(\overline{\Omega'})$ (see Subsection 2.7). Since for a.e. $t \in [0, \tau]$,

$$\text{supp}(\dot{p}(t)) = \text{supp}(\dot{p}(t)) \subset \Omega,$$

and by definition of $H(\dot{p}(t))$ (see (8)), the support of $H(\dot{p}(t))$ is also a subset of Ω . Hence, we have

$$H(\dot{p}(t))(\overline{\Omega'} \setminus \overline{\Omega}) = 0,$$

and consequently

$$\begin{aligned} H(\dot{p}(t))(\overline{\Omega'}) &= H(\dot{p}(t))(\overline{\Omega}) + H(\dot{p}(t))(\overline{\Omega'} \setminus \overline{\Omega}) \\ &= H(\dot{p}(t))(\overline{\Omega}) = H(\dot{p}(t))(\overline{\Omega}) < +\infty. \end{aligned} \quad (28)$$

Remark 2.7 ensures that

$$[\bar{\sigma}(t) : \dot{p}(t)] \leq H(\dot{p}(t)) \quad \text{in } \mathcal{M}(\overline{\Omega'}). \quad (29)$$

Due to the definition of $[\bar{\sigma}(t) : \dot{p}(t)]$ and using (27), we have

$$[\sigma(t) : \dot{p}(t)](\overline{\Omega}) = [\bar{\sigma}(t) : \dot{p}(t)](\overline{\Omega'}). \quad (30)$$

By (28) and (30), we get that

$$H(\dot{p}(t))(\overline{\Omega'}) = H(\dot{p}(t))(\overline{\Omega}) = [\sigma(t) : \dot{p}(t)](\overline{\Omega}) = [\bar{\sigma}(t) : \dot{p}(t)](\overline{\Omega'}). \quad (31)$$

Gathering (29) and (31), we obtain for a.e. $t \in [0, \tau]$,

$$H(\dot{p}(t)) = [\bar{\sigma}(t) : \dot{p}(t)] \quad \text{in } \mathcal{M}(\overline{\Omega'}). \quad (32)$$

Step 2: Spatial translation. For every $h \in \mathbb{R}^n$ be such that $|h| < \delta$, we define the translation operator τ_h of a generic function $F : (0, \tau) \times \Omega' \rightarrow X$ (where X is an Euclidean space) by

$$\tau_h F(t, x) = F(t, x + h) \quad \text{for all } (t, x) \in (0, \tau) \times \Omega.$$

Note that $\tau_h F$ is a function from $(0, \tau) \times \Omega$ to X .

As in the previous step, we check that the translated functions $\tau_h \bar{u}$, $\tau_h \bar{e}$, $\tau_h \bar{\sigma}$ and $\tau_h \bar{f}$ have the same regularity as the extended functions. First, observe that the translated source term $\tau_h \bar{f}$ is in $H^1([0, \tau]; L^2(\Omega; \mathbb{R}^n))$, the translated displacement $\tau_h \bar{u}$ in $W^{2,\infty}([0, \tau]; L^2(\Omega; \mathbb{R}^n))$ and the translated elastic strain $\tau_h \bar{e}$ belongs to $W^{1,\infty}([0, \tau]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$. We also define $\tau_h \bar{\sigma} := \mathbb{C} \tau_h \bar{e} \in W^{1,\infty}([0, \tau]; L^2(\Omega; K))$.

It remains to show that $\tau_h \bar{u} \in AC([0, \tau]; BD(\Omega))$. To do so, observe that for all $t \in [0, \tau]$ and all $\varphi \in \mathcal{C}_c^\infty(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, one has thanks to a change of variable

$$\int_{\Omega} \tau_h \bar{u}(t, x) \odot \nabla \varphi(x) \, dx = \int_{\Omega+h} \bar{u}(t, y) \odot \nabla \varphi(y - h) \, dx.$$

Using (24), it yields

$$\int_{\Omega} \tau_h \bar{u}(t, x) \odot \nabla \varphi(x) \, dx = \int_{\Omega} \bar{u}(t, y) \odot \nabla \varphi(y - h) \, dx. \quad (33)$$

The integration by parts formula in $BD(\Omega)$ (see [2, Theorem 3.2]) and the fact that the trace of $\bar{u}$ is null on $\partial\Omega$ imply that

$$\int_{\Omega} \bar{u}(t, y) \odot \nabla \varphi(y - h) dx. = - \int_{\Omega} \varphi(\cdot - h) dE\bar{u}(t). \quad (34)$$

Then, the definition of the push-forward of the measure $E\bar{u}(t)$ by the mapping $x \mapsto x - h$, denoted by $\tau_{-h\#}E\bar{u}(t)$, tells us that

$$- \int_{\Omega} \varphi(\cdot - h) dE\bar{u}(t) = - \int_{\Omega} \varphi d\tau_{-h\#}E\bar{u}(t). \quad (35)$$

Combining (33), (34) and (35), we obtain that for all $t \in [0, \tau]$,

$$E(\tau_h \bar{u})(t) = \tau_{-h\#}E\bar{u}(t) \quad \text{in } \mathcal{M}(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}). \quad (36)$$

It implies that $\tau_h \bar{u} \in AC([0, \tau]; BD(\Omega))$.

Finally, we show that the translated functions $\tau_h \bar{u}$ and $\tau_h \bar{e}$, and the push-forward measure $\tau_{-h\#}\bar{p}(t) \in \mathcal{M}(\bar{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})$ satisfy the additive decomposition in Ω and the boundary condition associated to the null (translated) boundary source term. Thanks to (25) and (36), the push-forward measure $\tau_{-h\#}\bar{p}(t) \in \mathcal{M}(\bar{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})$ satisfies for all $t \in [0, \tau]$

$$\tau_{-h\#}\bar{p}(t) = E\tau_h \bar{u}(t) - \tau_h \bar{e}(t) \quad \text{in } \Omega, \quad (37)$$

and due to the fact that for all $t \in [0, \tau]$, the measure $\bar{p}(t)$ is null on an h -neighborhood of $\partial\Omega$, we also have for all $t \in [0, \tau]$

$$\tau_{-h\#}\bar{p}(t) = 0 \odot \nu d\mathcal{H}^{n-1}. \quad (38)$$

These two previous points (37) and (38) and the regularity of $\tau_h \bar{u}$ and $\tau_h \bar{e}$ ensure that $\tau_{-h\#}\bar{p} \in AC([0, \tau]; \mathcal{M}(\bar{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n}))$.

Step 3 : The translation of the solution is a solution. We define the translation of the solution $(u_h, e_h, \sigma_h, p_h) := (\tau_h \bar{u}, \tau_h \bar{e}, \tau_h \bar{\sigma}, \tau_{-h\#}\bar{p})$, the translated source term $f_h := \tau_h \bar{f}$, the translated null boundary source term $w_h := 0$ and finally the translated initial data $(\tau_h u_0, \tau_h v_0, \tau_h e_0, \tau_h \sigma_0, \tau_h p_0) := (\tau_h \bar{u}_0, \tau_h \bar{v}_0, \tau_h \bar{e}_0, \tau_h \bar{\sigma}_0, \tau_{-h\#}\bar{p}_0)$.

Our goal is to prove that the quadruple $(u_h, e_h, \sigma_h, p_h)$ is the solution associated to the initial data $(\tau_h u_0, \tau_h v_0, \tau_h e_0, \tau_h \sigma_0, \tau_h p_0)$ and the source terms (f_h, w_h) .

First, we make sure that this initial data $(\tau_h u_0, \tau_h v_0, \tau_h e_0, \tau_h \sigma_0, \tau_h p_0)$ and the source terms f_h and w_h satisfy the hypotheses of [4, Theorem 5.1] (recalled in Section 2.8).

Indeed, since we assume that (10) is satisfied and also since we extend the source term f and the initial data u_0, v_0, e_0, σ_0 and p_0 by zero outside of Ω , we have, by the same reasoning as in the previous step, $\tau_h u_0 \in BD(\Omega) \cap$

$L^2(\Omega; \mathbb{R}^n)$, $\tau_h e_0 \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, $\tau_h p_0 = \tau_{-h\#} \bar{p}_0 \in \mathcal{M}(\bar{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})$ and $\tau_h v_0 \in H^1(\Omega; \mathbb{R}^n)$ satisfying the additive decomposition

$$E\tau_h u_0 = \tau_h e_0 + \tau_h p_0 \quad \text{in } \Omega,$$

and

$$\tau_h p_0 = 0 \odot \nu \mathcal{H}^{n-1} \quad \text{on } \partial\Omega. \quad (39)$$

Moreover, the translated initial data satisfies the compatibility conditions

$$\tau_h v_0 = 0 \quad \mathcal{H}^{n-1} - \text{a.e on } \partial\Omega \text{ and } -\text{div} \tau_h \sigma_0 = \tau_h f(0), \quad \text{a.e in } \Omega,$$

where $\tau_h \sigma_0 := \mathbb{C} \tau_h e_0 \in K$. To prove that the translated initial data satisfies the boundary condition

$$\tau_h p_0 = (w_h(0) - \tau_h u_0) \odot \nu \mathcal{H}^{n-1} = -\tau_h u_0 \odot \nu \mathcal{H}^{n-1} \quad \text{on } \partial\Omega,$$

we only need to prove that the trace of $\tau_h u_0$ is null on $\partial\Omega$, since we already know that $\tau_h p_0$ satisfies (39). But, due to the definition of δ (and since $|h| < \delta$), since u_0 satisfies (10) and since the trace of a function in $BD(\Omega)$ can be recovered by taking limits of averages of this function on balls centered on the boundary (see [2, Proposition 3.5]), we obtain that the trace of $\tau_h u_0$ is null on $\partial\Omega$ and as a consequence that the translated initial data satisfies the boundary condition. Lastly, we remark that we already know (from the previous step) that $f_h \in H^1([0, \tau]; L^2(\Omega; \mathbb{R}^n)) \subset AC([0, \tau]; L^2(\Omega; \mathbb{R}^n))$.

Now that we know that $(\tau_h u_0, \tau_h v_0, \tau_h e_0, \tau_h \sigma_0, \tau_h p_0)$ and the source terms (f_h, w_h) satisfy the hypotheses of Theorem 2.8, let us show that $(u_h, e_h, \sigma_h, p_h)$ is a solution of the dynamical elasto-plastic problem associated to these data, *i.e.* $(u_h, e_h, \sigma_h, p_h)$ satisfies the following conditions

(1) Regularity properties: we have

$$\begin{cases} u_h \in AC([0, \tau]; BD(\Omega)) \cap W^{2,\infty}([0, \tau]; L^2(\Omega; \mathbb{R}^n)), \\ e_h, \sigma_h \in W^{1,\infty}([0, \tau]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})), \\ p_h \in AC([0, \tau]; \mathcal{M}(\bar{\Omega}; \mathbb{M}_{\text{sym}}^{n \times n})), \end{cases}$$

(2) Equation of motion: $\ddot{u}_h - \text{div} \sigma_h = f_h$ a.e in $(0, \tau) \times \Omega$,

(3) Additive decomposition: for every $t \in [0, \tau]$,

$$Eu_h(t) = e_h(t) + p_h(t) \quad \text{in } \Omega,$$

(4) Stress constraint: for every $t \in [0, \tau]$, $\sigma_h(t) \in K$ a.e. in Ω with $\sigma_h(t) = \mathbb{C} e_h(t)$,

(5) (Homogeneous) Boundary condition: for every $t \in [0, \tau]$,

$$p_h(t) = -u_h(t) \odot \nu \mathcal{H}^{n-1} \quad \text{on } \partial\Omega,$$

(6) Flow rule: for a.e. $t \in [0, \tau]$, the distribution $[\sigma_h(t) : \dot{p}_h(t)]$ is a measure in $\mathcal{M}(\bar{\Omega})$ satisfying

$$H(\dot{p}_h(t)) = [\sigma_h(t) : \dot{p}_h(t)] \quad \text{in } \mathcal{M}(\bar{\Omega}),$$

(7) Initial conditions:

$$u_h(0) = \tau_h u_0, \quad \dot{u}_h(0) = \tau_h v_0, \quad \sigma_h(0) = \tau_h \sigma_0, \quad p_h(0) = \tau_{-h\#} p_0.$$

- Item 1 has already been proved in step 2.
- Now, we explain how to derive item 2: Using the extension on Ω' , we know that

$$\ddot{u} - \operatorname{div} \bar{\sigma} = \bar{f}, \quad \text{a.e. in } (0, \tau) \times \Omega'.$$

From which, we deduce the equation of motion by definition of the translation operator.

- Equation (37) is equivalent to item 3 and has already been obtained in the step 2.
- The stress constraint (item 4) is immediate from the definition of $\tau_h \bar{\sigma}$ in the step 2.
- Let us examine the point 5. Using the definition of u_h and the fact that the trace of a function in $BD(\Omega)$ can be recovered by taking limits of averages of this function on balls centered on the boundary (see [2, Proposition 3.5]), we obtain that for all $t \in [0, \tau]$

$$u_h(t) = 0 \quad \text{a.e. on } \partial\Omega. \quad (40)$$

Since in the step 2, we have already proven that (see Equation (38))

$$p_h = 0 \odot \nu \mathcal{H}^{n-1} \quad \text{on } \partial\Omega. \quad (41)$$

Using Equations (40) and (41), we get the boundary condition.

- We now turn to the flow rule (item 6). Since $(u_h, e_h, \sigma_h, p_h)(t)$ satisfies the hypotheses of Lemma 2.4 (thanks to items 1, 3 and 5), we can apply Definition 2.5 and we know that for a.e. $t \in [0, \tau]$ and for all $\varphi \in \mathcal{C}_0^\infty(\bar{\Omega})$,

$$\begin{aligned} \langle [\sigma_h(t) : \dot{p}_h(t)], \varphi \rangle &= - \int_{\Omega} \varphi \dot{u}_h(t) \cdot \operatorname{div} \sigma_h(t) \, dx \\ &\quad - \int_{\Omega} \sigma_h(t) : (\dot{u}_h(t) \odot \nabla \varphi) \, dx - \int_{\Omega} \sigma_h(t) : \dot{e}_h(t) \varphi \, dx. \end{aligned}$$

Using a change of variables and the fact that for all $t \in [0, \tau]$, the support of $\dot{u}(t)$ and $\bar{e}(t)$ is a subset of the compact set $\mathcal{X}^* \subset \Omega$, we get that

$$\begin{aligned} \langle [\dot{p}_h(t) : \sigma_h(t)], \varphi \rangle &= - \int_{\Omega} \varphi(\cdot - h) \dot{u}(t) \cdot \operatorname{div} \bar{\sigma}(t) \, dx \\ &\quad - \int_{\Omega} \bar{\sigma}(t) : (\dot{u}(t) \odot \nabla \varphi(\cdot - h)) \, dx - \int_{\Omega} \bar{\sigma}(t) : \dot{e}(t) \varphi(\cdot - h) \, dx, \end{aligned}$$

and again thanks to Definition 2.5, we obtain

$$\langle [\sigma_h(t) : \dot{p}_h(t)], \varphi \rangle = \langle [\bar{\sigma}(t) : \dot{p}(t)], \varphi(\cdot - h) \rangle = \langle H(\dot{p}(t)), \varphi(\cdot - h) \rangle, \quad (42)$$

where we used that $\varphi(\cdot - h) \in \mathcal{C}_0^\infty(\Omega')$ and $H(\dot{p}(t)) = [\bar{\sigma}(t) : \dot{p}(t)]$ in $\mathcal{M}(\bar{\Omega}')$ (according to (32)). To show item 6, it remains to prove that

$$\forall \varphi \in \mathcal{C}_0^\infty(\bar{\Omega}), \quad \langle H(\dot{p}(t)), \varphi(\cdot - h) \rangle = \langle H(\dot{p}_h(t)), \varphi \rangle,$$

which is equivalent, by definition of the push-forward measure, to

$$H(\dot{p}_h(t)) = \tau_{-h\#} H(\dot{p}(t)) \quad \text{in } \mathcal{M}(\bar{\Omega}). \quad (43)$$

In order to derive (43), we use (thanks to [15]) that for any Borel subset E of $\bar{\Omega}$

$$H(\dot{p}_h(t))(E) = \sup \left\{ \sum_{i=1}^N H(\dot{p}_h(t)(E_i)) \text{ with } N \in \mathbb{N} \text{ and } \bigsqcup_{i=1}^N E_i = E \right\},$$

where the supremum is taken over all pairwise disjoint partitions $\bigsqcup_i E_i$ of E into Borel sets. By definition of the push-forward measure, we obtain

$$H(\dot{p}_h(t))(E) = \sup \left\{ \sum_{i=1}^N H(\dot{p}(t)(E_i + h)) \text{ with } N \in \mathbb{N} \text{ and } \bigsqcup_{i=1}^N E_i = E \right\}.$$

Now since τ_h is a homeomorphism, one can associate to a pairwise disjoint partition of E into Borel sets a unique pairwise disjoint partition of $E + h$ into Borel sets. It implies that

$$\begin{aligned} H(\dot{p}_h(t))(E) &= \sup \left\{ \sum_{i=1}^N H(\dot{p}(t)(E_i + h)) \text{ with } N \in \mathbb{N} \text{ and } \bigsqcup_{i=1}^N E_i = E \right\} \\ &= H(\dot{p}(t))(E + h) = H(\dot{p}(t))((\tau_{-h})^{-1}(E)) = \tau_{-h\#} H(\dot{p}(t))(E). \end{aligned}$$

From which, we deduce (43) and it gives item 6.

- Due to the definition of τ_h , item 7 is automatically satisfied.

Since items 1–7 are satisfied, it shows that the translation of the solution is a solution (associated to translated data).

Step 4 : Regularity. Since $(u_h, e_h, \sigma_h, p_h)$ is a solution on $(0, \tau) \times \Omega$ associated to the initial condition $(\tau_h u_0, \tau_h v_0, \tau_h e_0, \tau_h \sigma_0, \tau_h p_0)$, the source term f_h and the null boundary source term, we can use the comparison principle (*i.e.* Proposition 3.2) applied to (u, e, σ, p) and $(u_h, e_h, \sigma_h, p_h)$ with the test function φ defined as

$$\varphi(t, x) = \begin{cases} 1 & \text{if } 0 \leq t \leq t_0, \\ \frac{t_0 + \varepsilon - t}{\varepsilon} & \text{if } t_0 < t < t_0 + \varepsilon, \\ 0 & \text{if } t_0 + \varepsilon \leq t \leq \tau, \end{cases} \quad (44)$$

with $t_0 \in [0, \tau)$ and $\varepsilon > 0$ such that $(t_0, t_0 + \varepsilon) \subset [0, \tau]$. Thus, Proposition 3.2 yields

$$\begin{aligned} & \frac{1}{\varepsilon} \left(\int_{t_0}^{t_0+\varepsilon} \int_{\Omega} |\dot{u}_h - \dot{u}|^2 dx dt + \int_{t_0}^{t_0+\varepsilon} \int_{\Omega} Q(e_h - e) dx dt \right) \\ & \leq \int_{\Omega} |v_0 - \tau_h v_0|^2 dx + \int_{\Omega} Q(e_0 - \tau_h e_0) dx + 2 \int_0^{\tau} \int_{\Omega} |f_h - f| |\dot{u}_h - \dot{u}| dx dt, \end{aligned}$$

since $0 \leq \varphi \leq 1$. Since $\dot{u}_h$ and $\dot{u}$ belong to $W^{1,\infty}([0, \tau]; L^2(\Omega; \mathbb{R}^n))$ and e_h and e belong to $W^{1,\infty}([0, \tau]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$, we can pass to the limit $\varepsilon \rightarrow 0^+$ in the previous inequality and we get

$$\begin{aligned} & \int_{\Omega} |\dot{u}_h(t_0) - \dot{u}(t_0)|^2 dx dt + \int_{\Omega} Q(e_h(t_0) - e(t_0)) dx dt \\ & \leq \int_{\Omega} |v_0 - \tau_h v_0|^2 dx + \int_{\Omega} Q(e_0 - \tau_h e_0) dx + 2 \int_0^{\tau} \int_{\Omega} |f_h - f| |\dot{u}_h - \dot{u}| dx dt, \end{aligned}$$

for every $t_0 \in [0, \tau)$.

According to Young's inequality, to (4) and (5), and since we have $v_0 \in H^1(\Omega; \mathbb{R}^n)$, $\sigma_0 \in H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, $f \in H^1((0, \tau) \times \Omega; \mathbb{R}^n)$, we get that for any $|h| < \delta$ the following inequality holds

$$\begin{aligned} & \text{ess sup}_{t \in [0, \tau]} \|(\dot{u}_h - \dot{u})(t)\|_{L^2(\Omega; \mathbb{R}^n)}^2 + \alpha_{\mathbb{C}} \text{ess sup}_{t \in [0, \tau]} \|(e_h - e)(t)\|_{L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})} \\ & \leq C(T^*) |h|^2 \left(\|Dv_0\|_{L^2(\Omega; \mathbb{M}^{n \times n})}^2 + \|D\sigma_0\|_{L^2(\Omega; \mathbb{R}^n \times \mathbb{M}^{n \times n})}^2 \right. \\ & \quad \left. + \int_0^{T^*} \|Df(t)\|_{L^2(\Omega; \mathbb{M}^{n \times n})}^2 dt \right), \end{aligned}$$

from which we get that $\dot{u} \in L^\infty(0, \tau; H^1(\Omega; \mathbb{R}^n))$. It also yields, due to the relation $\sigma = \mathbb{C}e$, $\sigma \in L^\infty(0, \tau; H^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$.

Step 5 : Additional regularity. In the following, we suppose that $p_0 \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ and $u_0 \in H^1(\Omega; \mathbb{R}^n)$. Writing for all $t \in [0, T]$,

$$u(t) = u_0 + \int_0^t \dot{u}(s) ds,$$

as a Bochner integral in $L^2(\Omega; \mathbb{R}^n)$, we obtain, thanks to the regularity of the velocity field derived in step 4, that $u \in W^{1,\infty}([0, \tau]; H^1(\Omega; \mathbb{R}^n))$.

Finally, due to Theorem 2.8, we know that $\sigma \in W^{1,\infty}([0, \tau]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$ and consequently, from the additive decomposition, we get $p = Eu - \sigma \in L^\infty(0, \tau; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$ and $\dot{p} = E\dot{u} - \dot{\sigma} \in L^\infty(0, \tau; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$ which gives the desired regularity for the plastic strain. $\square$

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