Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On the qualitative behavior of the solutions to second-order neutral delay differential equations

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Abstract

Differential equations of second order appear in numerous applications such as fluid dynamics, electromagnetism, quantum mechanics, neural networks and the field of time symmetric electrodynamics. The aim of this work is to establish necessary and sufficient conditions for the oscillation of the solutions to a second-order neutral differential equation. First, we have taken a single delay and later the results are generalized for multiple delays. Some examples are given and open problems are presented.

Keywords: Oscillation; Non-oscillation; Delay; Neutral; Lebesgue's Dominated Convergence theorem; Necessary and sufficient conditions

1 Introduction

Consider the class of nonlinear neutral delay differential equations of the form

$$\left(a\left(w'\right)^{\mu}\right)'(y) + c(y)g\left(u\left(\varsigma(y)\right)\right) = 0, \tag{1}$$

where $w(y) = u(y) + b(y)u(\vartheta(y))$ and μ is the ratio of two odd positive integers. We assume the following conditions hold.

(A1) $a, c, \vartheta, \varsigma \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $\vartheta(y) \le y$, $\varsigma(y) \le y$ for $y \ge y_0$, $\vartheta(y) \to \infty$, $\varsigma(y) \to \infty$ as $y \to \infty$.

(A2) $g \in C(\mathbb{R}, \mathbb{R})$ is non-decreasing and odd with $ug(u) > 0$ for $u \neq 0$.

(A3) $a(y) > 0$ and $\int_0^{\infty}(a(\eta))^{-1/\mu}\,d\eta = \infty$. By letting $A(y) = \int_0^{y}(a(\eta))^{-1/\mu}\,d\eta$, we have $\lim_{y\to\infty} A(y) = \infty$.

(A4) $b \in C(\mathbb{R}_+, \mathbb{R}_-)$ with $-1 + (2/3)^{1/\mu} \le -b_0 \le b(y) \le 0$ for $y \in \mathbb{R}_+$.

(A5) $b \in C(\mathbb{R}_+, \mathbb{R}_-)$ with $-1 < -b_0 \le b(y) \le 0$ for $y \in \mathbb{R}_+$.

In 1978, Brands [1] showed that the solutions to

$$u''(y) + c(y)u\left(y - \varsigma(y)\right) = 0$$

are oscillatory, if and only if, the solutions to $u''(y) + c(y)u(y) = 0$ are oscillatory. Baculikova *et al.* [2] considered (1) and studied the oscillatory behavior of (1) for $g(u) = u$,

$0 \le b(y) \le b_0 < \infty$ and (A3). They obtained sufficient conditions for the oscillation of the solutions of the linear counterpart of (1), using comparison techniques. Chatzarakis *et al.* [3] considered the equation

$$\bigl(a(u')^{\mu_2}\bigr)'(y) + c(y)u^{\mu_2}\bigl(\varsigma(y)\bigr) = 0. \tag{2}$$

Also, Chatzarakis *et al.* [4] studied (2) to obtain new oscillation criteria. Džurina [5] studied the linear counterpart of (1) when $0 \le b(y) \le b_0 < \infty$ and (A3) and established sufficient conditions for the oscillation of the solutions of the linear counterpart of (1) by comparison techniques. Karpuz *et al.* [6] studied (1) for various ranges of the neutral coefficient b. Pinelas and Santra [7] studied necessary and sufficient conditions for the solutions of

$$\bigl(u(y) + b(y)u(y-\vartheta)\bigr)' + \sum_{i=1}^{m} c_j(y)g\bigl(u(y-\varsigma_j)\bigr) = 0.$$

Wong [8] obtained necessary and sufficient conditions for the oscillation of

$$\bigl(u(y) + bu(y-\vartheta)\bigr)'' + c(y)g(y-\varsigma) = 0,$$

where the constant b satisfies $-1 < b < 0$. Grace et al. [9] studied (1) and established sufficient conditions for $0 \le b(y) < 1$. For further work on this type of equations, we refer the reader to [10–36] and the references cited therein. We may note that most of the authors considered only sufficient conditions, and only a few considered necessary and sufficient conditions. Hence, the objective of this work is to establish both necessary and sufficient conditions for oscillation of (1) without using comparison techniques.

In Sect. 2 some preliminary results are presented, Sect. 3 deals with main results, Sect. 4 represents the conclusion and the final section includes open problems.

2 Preliminary results

In this section, two lemmas are presented which we need for our work in the sequel.

Lemma 2.1 *Under the assumptions* (A1)–(A3) *and* (A4) *or* (A5) *and the solution u of* (1) *is an eventually positive solution, we have*

(i) $w(y) < 0$, $w'(y) > 0$ and $(a(w')^{\mu})'(y) < 0$;

(ii) $w(y) > 0$, $w'(y) > 0$ and $(a(w')^{\mu})'(y) < 0$,

for sufficiently large y.

Proof Assume there exists a $y_1 \ge y_0$ such that $u(y) > 0$, $u(\vartheta(y))$, and $u(\varsigma(y)) > 0$ for $y \ge y_1$. From (1) and (A2), we have

$$\bigl(a(w')^{\mu}\bigr)'(y) = -c(y)g\bigl(u\bigl(\varsigma(y)\bigr)\bigr) < 0 \quad \text{for } y \ge y_1, \tag{3}$$

which implies that $(a(w')^{\mu})(y)$ is non-increasing on $[y_1, \infty)$. We have $a(y) > 0$, and thus either $w'(y) < 0$ or $w'(y) > 0$ for $y \ge y_2$, where $y_2 \ge y_1$.

If $w'(y) > 0$ for $y \ge y_2$, then we have (i) and (ii). We prove now that $w'(y) < 0$ cannot occur.

If $w'(y) < 0$ for $y \geq y_2$, then there exists $\kappa_1 > 0$ such that $(a(w')^{\mu})(y) \leq -\kappa_1$ for $y \geq y_2$, which yields upon integration over $[y_2, y) \subset [y_2, \infty)$ after dividing through by a

$$w(y) \leq w(y_2) - \kappa_1^{1/\mu} \int_{y_2}^{y} \bigl(a(\eta)\bigr)^{-1/\mu}\, d\eta \quad \text{for } y \geq y_2. \tag{4}$$

By virtue of condition (A3), $\lim_{t\to\infty} w(y) = -\infty$. We consider the following possibilities:

Let the solution u be unbounded. There exists a sequence $\{y_k\}$ such that $\lim_{k\to\infty} y_k = \infty$ and $\lim_{k\to\infty} u(y_k) = \infty$, where $u(y_k) = \max\{u(\eta) : y_0 \leq \eta \leq y_k\}$. Since $\lim_{y\to\infty} \vartheta(y) = \infty$, $\vartheta(y_k) > y_0$ for all sufficiently large k. By $\vartheta(y) \leq y$,

$$u\bigl(\vartheta(y_k)\bigr) = \max\bigl\{u(\eta) : y_0 \leq \eta \leq \vartheta(y_k)\bigr\} \leq \max\bigl\{u(\eta) : y_0 \leq \eta \leq y_k\bigr\} = u(y_k).$$

Therefore, for all large k,

$$w(y_k) = u(y_k) + b(y_k)u\bigl(\vartheta(y_k)\bigr) \geq \bigl(1 + b(y_k)\bigr)u(y_k) > 0,$$

which contradicts $\lim_{y\to\infty} w(y) = -\infty$.

Let the solution u be bounded, then w is bounded, from which one concludes $\lim_{y\to\infty} w(y) = -\infty$, a contradiction. Hence, w satisfies one of the cases (i) or (ii). This completes the proof. □

Lemma 2.2 *Under the assumptions* (A1)–(A3), (A4) *or* (A5), (i) *and u is an eventually positive solution of* (1), *we have* $\lim_{y\to\infty} u(y) = 0$.

Proof Assume that there exists a $y_1 \geq y_0$ such that $u(y) > 0$, $u(\vartheta(y))$, and $u(\varsigma(y)) > 0$ for $y \geq y_1$. Then Lemma 2.1 holds and w satisfies one of the cases (i) or (ii) for $y_2 \geq y_1$, where $y \geq y_2$. Let w satisfy (i) for $y \geq y_2$. Therefore,

$$\begin{aligned} 0 &\geq \lim_{y\to\infty} w(y) = \limsup_{y\to\infty} w(y) \geq \limsup_{y\to\infty}\bigl(u(y) - b_0 u\bigl(\vartheta(y)\bigr)\bigr) \\ &\geq \limsup_{y\to\infty} u(y) + \liminf_{t\to\infty}\bigl(-b_0 u\bigl(\vartheta(y)\bigr)\bigr) = (1 - b_0)\limsup_{y\to\infty} u(y), \end{aligned}$$

which implies that $\limsup_{y\to\infty} u(y) = 0$ and hence $\lim_{y\to\infty} u(y) = 0$. □

Remark 1 In view of (ii) of Lemma 2.1, it is obvious that $\lim_{y\to\infty} w(y) > 0$, i.e., there exists $\kappa_1 > 0$ such that $w(y) \geq \kappa_1$ for all large y.

3 Main results

In this section, we establish the necessary and sufficient conditions for the oscillation of the solution of (1) by considering the two cases when $g(v)/v^{\mu_1}$ is non-increasing and $g(v)/v^{\mu_1}$ is non-decreasing.

3.1 The case when $g(v)/v^{\mu_1}$ is non-increasing

Suppose that there exists μ_1 such that $0 < \mu_1 < \mu$ and

$$\frac{g(v)}{v^{\mu_1}} \geq \frac{g(u)}{u^{\mu_1}} \quad \text{for } 0 < v \leq u. \tag{5}$$

For example the function $g(u) = |u|^{\mu_2}\operatorname{sgn}(u)$ with $0 < \mu_2 < \mu_1 < \mu$ satisfying (5).

Theorem 3.1 *Assume that* (A1)–(A4) *and* (5) *hold. Then each unbounded solution of* (1) *is oscillatory if and only if*

$$\int_Y^\infty c(\eta)g\bigl(\kappa^{1/\mu}A\bigl(\varsigma(\eta)\bigr)\bigr)\,d\eta = +\infty \quad \forall Y > 0 \text{ and } \kappa > 0. \tag{6}$$

Proof On the contrary, we assume that there exists a nonoscillatory unbounded solution $u(y)$ of (1). Suppose that the solution $u(y)$ is eventually positive. Then there exists $y_1 \geq y_0$ such that $u(y) > 0$, $u(y) > 0$, $u(\vartheta(y)) > 0$ and $u(\varsigma(y)) > 0$ for $y \geq y_1$. Proceeding as in the proof of Lemma 2.1, we see that $(a(w')^\mu)(y)$ is non-increasing, and w satisfies one of the cases (i) or (ii) on $[y_2,\infty)$, where $y_2 \geq y_1$. Then we have the following two possible cases.

Case 1. Let w satisfy (i) for $y \geq y_2$. As u is the unbounded solution, there exists $y \geq y_2$ such that $u(y) = \max\{u(s) : y_2 \leq s \leq T\}$. Since $w(y) = u(y) + b(y)u(\vartheta(y))$, we have $u(y) \leq w(y) + \{1 - (2/3)^{1/\mu}\}u(\vartheta(y)) < u(y)$, which leads a contradiction.

Case 2. Let w satisfy (ii) for $y \geq y_2$. Note that $\lim_{y\to\infty}(a(w')^\mu)(y)$ exists. Using $w(y) \leq u(y)$ in (1) and integrating the new inequality from y to $+\infty$, we obtain

$$\int_y^\infty c(\eta)g\bigl(w\bigl(\varsigma(\eta)\bigr)\bigr)\,d\eta \leq \bigl(a\bigl(w'\bigr)^\mu\bigr)(y).$$

That is,

$$w'(y) \geq \left[\frac{1}{a(y)}\int_y^\infty c(\eta)g\bigl(w\bigl(\varsigma(\eta)\bigr)\bigr)\,d\eta\right]^{1/\mu} \tag{7}$$

for $y \geq y_3$. Let $y_4 > y_3$ be a point such that

$$A(y) - A(y_3) \geq \frac{1}{2}A(y), \quad y \geq y_4.$$

Then integrating (7) from y_3 to y, we get

$$\begin{aligned} w(y) - w(y_3) &\geq \int_{y_3}^y \left[\frac{1}{a(\eta)}\int_\eta^\infty c(\zeta)g\bigl(w\bigl(\varsigma(\zeta)\bigr)\bigr)\,d\zeta\right]^{1/\mu} d\eta \\ &\geq \int_{y_3}^y \left[\frac{1}{a(\eta)}\int_y^\infty c(\zeta)g\bigl(w\bigl(\varsigma(\zeta)\bigr)\bigr)\,d\zeta\right]^{1/\mu} d\eta, \end{aligned}$$

i.e.,

$$\begin{aligned} w(y) &\geq \bigl(A(y) - A(y_3)\bigr)\left[\int_y^\infty c(\zeta)g\bigl(w\bigl(\varsigma(\zeta)\bigr)\bigr)\,d\zeta\right]^{1/\mu} \\ &\geq \frac{1}{2}A(y)\left[\int_y^\infty c(\zeta)g\bigl(w\bigl(\varsigma(\zeta)\bigr)\bigr)\,d\zeta\right]^{1/\mu}. \end{aligned} \tag{8}$$

Since $(a(w')^\mu)(y)$ is non-increasing on $[y_4,\infty)$, there exist $\kappa > 0$ and $y_5 > y_4$ such that $(a(w')^\mu)(y) \leq \kappa$ for $y \geq y_5$. Integrating the inequality $w'(y) \leq (\kappa/a(y))^{1/\mu}$, we have

$$w(y) \leq w(y_5) + \kappa^{1/\mu}\bigl(A(y) - A(y_5)\bigr).$$

Since $\lim_{t\to\infty} A(y) = \infty$, the last inequality becomes

$$w(y) \le \kappa^{1/\mu} A(y) \quad \text{for } y \ge y_5.$$

On the other hand, (5) implies that

$$g\bigl(w\bigl(\varsigma(\zeta)\bigr)\bigr) = \frac{g(w(\varsigma(\zeta)))}{w^{\mu_1}(\varsigma(\zeta))} w^{\mu_1}\bigl(\varsigma(\zeta)\bigr) \ge \frac{g(\kappa^{1/\mu} A(\varsigma(\zeta)))}{(\kappa^{1/\mu} A(\varsigma(\zeta)))^{\mu_1}} w^{\mu_1}\bigl(\varsigma(\zeta)\bigr).$$

Consequently, (8) becomes

$$w(y) \ge \frac{A(y)}{2}\left[\int_y^\infty \frac{c(\zeta) g(\kappa^{1/\mu} A(\varsigma(\zeta))) w^{\mu_1}(\varsigma(\zeta))}{(\kappa^{1/\mu} A(\varsigma(\zeta)))^{\mu_1}}\, d\zeta\right]^{1/\mu}.$$

If we define

$$\Upsilon(y) = \int_y^\infty \frac{c(\zeta) g(\kappa^{1/\mu} A(\varsigma(\zeta))) w^{\mu_1}(\varsigma(\zeta))}{(\kappa^{1/\mu} A(\varsigma(\zeta)))^{\mu_1}}\, d\zeta,$$

then $w^{\mu_1}/(\kappa^{1/\mu} A)^{\mu_1} \ge \Upsilon^{\mu_1/\mu}/(2\kappa^{1/\mu})^{\mu_1}$. Taking the derivative of Υ we get

$$\Upsilon'(y) \le -\frac{g(\kappa^{1/\mu} A(\varsigma(y))) c(y) w^{\mu_1}(\varsigma(y))}{(\kappa^{1/\mu} A(\varsigma(y)))^{\mu_1}} \le -\frac{c(y) g(\kappa^{1/\mu} A(\varsigma(y)))}{(2\kappa^{1/\mu})^{\mu_1}} \Upsilon^{\mu_1/\mu}\bigl(\varsigma(y)\bigr) \le 0.$$

Therefore, $\Upsilon(y)$ is non-increasing on $[y_5, \infty)$ so $\Upsilon^{\mu_1/\mu}(\varsigma(y))/\Upsilon^{\mu_1/\mu}(y) \ge 1$, and

$$\begin{aligned}\bigl(\Upsilon^{1-\mu_1/\mu}(y)\bigr)' &\le -(1-\mu_1/\mu)\Upsilon^{-\mu_1/\mu}(y)\frac{c(y) g(\kappa^{1/\mu} A(\varsigma(y)))}{(2\kappa^{1/\mu})^{\mu_1}} \Upsilon^{\mu_1/\mu}\bigl(\varsigma(y)\bigr) \\ &\le -(1-\mu_1/\mu)\frac{c(y) g(\kappa^{1/\mu} A(\varsigma(y)))}{(2\kappa^{1/\mu})^{\mu_1}}.\end{aligned}$$

We have $\mu_1/\mu < 1$ and $\Upsilon(y)$ is positive and non-increasing. Integrating the last inequality, from y_5 to y, we have

$$\frac{(1-\mu_1/\mu)}{(2\kappa^{1/\mu})^{\mu_1}} \int_{t^5}^y c(\eta) g\bigl(\kappa^{1/\mu} A\bigl(\varsigma(\eta)\bigr)\bigr)\, d\eta \le -\bigl[\Upsilon^{1-\mu_1/\mu}(\eta)\bigr]_{y_5}^y < \Upsilon^{1-\mu_1/\mu}(y_5) < \infty,$$

which contradicts (6).

If $u(y) < 0$ for $y \ge y_1$, then we set $y(y) := -u(y)$ for $y \ge y_1$ in (1). Using (A2), we find

$$\bigl(a(y)\bigl(\overline{w}'(y)\bigr)^\mu\bigr) + c(y)\overline{g}\bigl(y\bigl(\varsigma(y)\bigr)\bigr) = 0 \quad \text{for } y \ge y_1,$$

where $\overline{w}(y) = y(y) + b(y)y(\vartheta(y))$ and $\overline{g}(u) := -g(-u)$ for $u \in \mathbb{R}$. Clearly, $\overline{g}$ satisfies (A2). Then, proceeding as above, we can find the same contradiction.

To prove the condition (6) is necessary, assume that (6) does not hold; so for some $\kappa > 0$ and $y \ge y_0$ we have

$$\int_Y^\infty c(\eta) g\bigl(\kappa^{1/\mu} A\bigl(\varsigma(\eta)\bigr)\bigr)\, d\eta \le \frac{\kappa}{3}.$$

We set

$$S = \Big\{u : u \in C\big([y_0,\infty),\mathbb{R}\big), u(y) = 0 \text{ for } y \in [y_0, Y] \text{ and}$$
$$\left(\frac{\kappa}{3}\right)^{1/\mu}\big[A(y) - A(Y)\big] \le u(y) \le \kappa^{1/\mu}\big[A(y) - A(Y)\big] \text{ for } y \ge y_0\Big\}.$$

We define the operator $\Omega : S \to C([y_0,+\infty),\mathbb{R})$ by

$$(\Omega u)(y) = \begin{cases} 0, & y \in [y_0, Y], \\ -b(y)u(\vartheta(y)) + \int_Y^y [\frac{1}{a(\eta)}[\frac{\kappa}{3} + \int_\eta^\infty c(\zeta)g(u(\varsigma(\zeta)))\,d\zeta]]^{1/\mu}\,d\eta, & y \ge Y. \end{cases}$$

For every $u \in S$ and $y \ge Y$, we have

$$\begin{aligned} (\Omega u)(y) &\ge \int_Y^y \left[\frac{1}{a(\eta)}\left[\frac{\kappa}{3} + \int_\eta^\infty c(\zeta)g\big(u\big(\varsigma(\zeta)\big)\big)\,d\zeta\right]\right]^{1/\mu} d\eta \\ &\ge \int_Y^y \left[\frac{1}{a(\eta)}\frac{\kappa}{3}\right]^{1/\mu} d\eta = \left(\frac{\kappa}{3}\right)^{1/\mu}\big[A(y) - A(Y)\big]. \end{aligned}$$

For every $u \in S$ and $y \ge Y$, we have $u(y) \le \kappa^{1/\mu}A(y)$ and $g(u(y)) \le g(\kappa^{1/\mu}A(y))$. Then

$$\begin{aligned} (\Omega u)(y) &\le -b(y)u\big(\vartheta(y)\big) + \int_T^y \left[\frac{1}{a(\eta)}\left(\frac{\kappa}{3} + \frac{\kappa}{3}\right)\right]^{1/\mu} d\eta \\ &\le b_0\kappa^{1/\mu}\big[A\big(\vartheta(y)\big) - A(Y)\big] + (2\kappa/3)^{1/\mu}\big[A(y) - A(Y)\big] \\ &\le b_0\kappa^{1/\mu}\big[A(y) - A(Y)\big] + (2\kappa/3)^{1/\mu}\big[A(y) - A(Y)\big] \\ &= \big(b_0 + (2/3)^{1/\mu}\big)\kappa^{1/\mu}\big[A(y) - A(Y)\big] \le \kappa^{1/\mu}\big[A(y) - A(Y)\big], \end{aligned}$$

which implies that $(\Omega u)(y) \in S$. Let us define now a sequence of continuous function $v_n : [y_0,+\infty) \to \mathbb{R}$ by the recursive formula

$$\begin{aligned} u_0(y) &= \begin{cases} 0, & y \in [y_0, Y], \\ \frac{\kappa}{3}[A(y) - A(Y)], & y \ge Y, \end{cases} \\ u_n(y) &= (\Omega u_{n-1})(y), \quad n \ge 1. \end{aligned}$$

Inductively, it is easy to verify that, for $n > 1$,

$$\left(\frac{\kappa}{3}\right)^{1/\mu}\big[A(y) - A(Y)\big] \le u_{n-1}(y) \le u_n(y) \le \kappa^{1/\mu}\big[A(y) - A(Y)\big].$$

Therefore the point-wise limit of the sequence exists. Let $\lim_{y\to\infty} u_n(y) = v(y)$ for $y \ge y_0$. By Lebesgue's dominated convergence theorem, $u \in S$ and $(\Omega u)(y) = u(y)$, where $u(y)$ is a solution of (1) on $[Y,\infty)$ such that $u(y) > 0$. Hence, (6) is necessary. This completes the proof. □

Example 3.2 Consider the delay differential equation

$$\left(e^{-y}\left(\left(u(y)-e^{-y}u(y-1)\right)'\right)^{3/5}\right)'+y\left(u(y-2)\right)^{1/3}=0,\quad y\geq 0. \tag{9}$$

Here $\mu=3/5$, $a(y)=e^{-y}$, $-1<b(y)=-e^{-y}\leq 0$, $\vartheta(y)=y-1$, $\varsigma(y)=y-2$, $A(y)=\int_0^y e^{5s/3}\,ds=\frac{3}{5}(e^{5y/3}-1)$, $g(\nu)=\nu^{1/3}$. For $\mu_1=1/2$, we have a decreasing function $g(\nu)/\nu^{\mu_1}=\nu^{-1/6}$. Now

$$\int_0^\infty c(\eta)g\left(\kappa^{1/\mu}A\left(\varsigma(\eta)\right)\right)d\eta=\int_0^\infty \eta\left(\kappa^{5/3}\frac{3}{5}\left(e^{5(\eta-2/3}-1\right)\right)^{1/3}d\eta=\infty\quad\forall\kappa>0.$$

So, all the conditions of Theorem 3.1 hold, and therefore every unbounded solution of (9) is oscillatory.

Theorem 3.3 *Let assumptions* (A1)–(A4) *hold. Then each unbounded solution of* (1) *oscillates if and only if* (6) *holds for every* $\kappa>0$.

Proof To prove sufficiency by contradiction, assume that the solution u of (1) is eventually positive and unbounded. So, there exists $y_1\geq y_0$ such that $u(y)>0$, $u(\vartheta(y))>0$ and $u(\varsigma(y))>0$ for $y\geq y_1$. Proceeding as in the proof of Lemma 2.1, $(a(w')^\mu)(y)$ is non-increasing, w satisfies one of the cases (i) or (ii) on $[y_2,\infty)$, where $y_2\geq y_1$. We have the following two possible cases.

Case 1. Let w satisfy (i) for $y\geq y_2$. This case is similar to the proof of Theorem 3.1.

Case 2. Let w satisfy (ii) for $y\geq y_2$. Since $w(y)$ is unbounded and monotonically increasing, it follows that

$$\lim_{y\to\infty}\frac{w^\mu(y)}{A^\mu(y)}=\lim_{y\to\infty}\frac{(w'(y))^\mu}{(A'(y))^\mu}=\lim_{y\to\infty}\left(a\left(w'\right)^\mu\right)(y)=c<\infty.$$

If $c=0$, then $\lim_{t\to\infty}A(y)=+\infty$ implies that $\lim_{t\to\infty}w(y)<+\infty$, which is invalid ($\because w(y)$ is unbounded). Hence $c\neq 0$. Therefore, there exist a constant $\kappa>0$ and a $y_2>y_1$ such that $w(y)\geq\kappa^{1/\mu}A(y)$ for $y\geq y_2$. Consequently, $u(y)\geq w(y)\geq\kappa^{1/\mu}A(y)$ for $y\geq y_2$. Using $u(y)\geq\kappa^{1/\mu}A(y)$ in (1) and then integrating the final inequality from y_2 to $+\infty$, we obtain a contradiction to (6) for every $\kappa>0$.

By using the same transformation as in the proof of Theorem 3.1 we can get a contradiction for an eventually negative unbounded solution, so we omit it here.

One can prove the necessary part by following the proof of Theorem 3.1. So we omit it here. The proof of the theorem is complete. □

Theorem 3.4 *Assume that* (A1)–(A4) *and* (5) *hold. Then each solution of* (1) *is oscillatory or* $\lim_{y\to\infty}u(y)=0$ *if and only if* (6) *holds for every* $\kappa>0$.

Proof On the contrary, we assume that the solution u of (1) is eventually positive. Then there exists $y_1\geq y_0$ such that $u(y)>0$, $u(\vartheta(y))>0$ and $u(\varsigma(y))>0$ for $y\geq y_1$. Proceeding as in the proof of Lemma 2.1, we see $(a(w')^\mu)(y)$ is non-increasing, and w satisfies one of the cases (i) or (ii) on $[y_2,\infty)$, where $y_2\geq y_1$. Thus, we have the following two possible cases.

Case 1. Let w satisfy (i) for $y\geq y_2$. Then, by Lemma 2.2, we have $\lim_{y\to\infty}u(y)=0$.

Case 2. Let w satisfy (ii) for $y\geq y_2$. The case follows from the proof of Theorem 3.1.

The necessary part is similar to Theorem 3.1. The proof of the theorem is complete. □

3.2 The case when $g(u)/u^{\mu_1}$ is non-decreasing

Suppose that there exists $\mu_1 > \mu$ such that

$$\frac{g(v)}{v^{\mu_1}} \leq \frac{g(u)}{u^{\mu_1}} \quad \text{for } 0 < v \leq u. \tag{10}$$

For example we might consider the function $g(u) = |u|^{\mu_2} \operatorname{sgn}(u)$ with $\mu < \mu_1 < \mu_2$ satisfying (10).

Theorem 3.5 *Assume that* (A1)–(A3), (A5), (10), $\varsigma'(y) \geq 1$ *hold. Then each solution of* (1) *oscillates or* $\lim_{y\to\infty} u(y) = 0$ *if and only if*

$$\int_Y^\infty \left[\frac{1}{a(\zeta)}\left[\int_\zeta^\infty c(\eta)\,d\eta\right]\right]^{1/\mu} d\zeta = +\infty \quad \forall y > 0. \tag{11}$$

Proof Proceeding in the proof of Theorem 3.4, we can conclude that $\lim_{y\to\infty} u(y) = 0$ when z satisfies (i). Let us consider *Case* 2, for $y \geq y_2$. By Remark 1, there exist a constant $\kappa > 0$ and $y_2 > y_1$ such that $z(\varsigma(y)) \geq \kappa$ for $y \geq y_2$. Consequently,

$$g\bigl(w\bigl(\varsigma(y)\bigr)\bigr) = \frac{g(w(\varsigma(y)))}{w^{\mu_1}(\varsigma(y))} w^{\mu_1}\bigl(\varsigma(y)\bigr) \geq \frac{g(\kappa)}{\kappa^{\mu_1}} w^{\mu_1}\bigl(\varsigma(y)\bigr) \tag{12}$$

for $y \geq y_2$. Using $w(y) \leq u(x)$ and (12) in (1), and then integrating the final inequality we have

$$\lim_{A\to\infty}\bigl[\bigl(a\bigl(w'\bigr)'\bigr)(\eta)\bigr]_y^A + \frac{g(\kappa)}{\kappa^{\mu_1}} \int_y^\infty c(\zeta) w^{\mu_1}\bigl(\varsigma(\zeta)\bigr)\,d\zeta \leq 0.$$

Since $(a(w')')(y)$ is non-increasing and positive, we have

$$\frac{g(\kappa)}{\kappa^{\mu_1}} \int_y^\infty c(\eta) w^{\mu_1}\bigl(\varsigma(\eta)\bigr)\,d\eta \leq \bigl(a\bigl(w'\bigr)^\mu\bigr)(y) \leq \bigl(a\bigl(w'\bigr)^\mu\bigr)\bigl(\varsigma(y)\bigr) \leq a(y)\bigl(\bigl(w'\bigr)^\mu\bigr)\bigl(\varsigma(y)\bigr)$$

for all $y \geq y_2$. Therefore,

$$\left(\frac{g(\kappa)}{\kappa^{\mu_1}}\right)^{1/\mu} \left[\frac{1}{a(y)}\left[\int_y^\infty c(\zeta) w^{\mu_1}\bigl(\varsigma(\zeta)\bigr)\,d\zeta\right]\right]^{1/\mu} \leq w'\bigl(\varsigma(y)\bigr)$$

implies that

$$\left(\frac{g(\kappa)}{\kappa^{\mu_1}}\right)^{1/\mu} \left[\frac{1}{a(y)}\left[\int_y^\infty c(\zeta)\,d\zeta\right]\right]^{1/\mu} \leq \frac{w'(\varsigma(y))}{w^{\mu_1/\mu}(\varsigma(y))} \leq \frac{w'(\varsigma(y))\varsigma'(y)}{w^{\mu_1/\mu}(\varsigma(y))}.$$

Integrating the final inequality from y_2 to $+\infty$, we have

$$\begin{aligned}\left(\frac{g(\kappa)}{\kappa^{\mu_1}}\right)^{1/\mu} \int_{y_2}^\infty \left[\frac{1}{a(\zeta)}\left[\int_\zeta^\infty c(\eta)\,d\eta\right]\right]^{1/\mu} d\zeta &< \int_{y_2}^\infty \frac{w'(\varsigma(\eta))\varsigma'(\eta)}{w^{\mu_1/\mu}(\varsigma(\eta))}\,d\eta \\ &\leq \frac{w^{1-\mu_1/\mu}(\varsigma(y_2))}{\mu_1/\mu - 1} < \infty,\end{aligned}$$

which contradicts (11).

Next, we show that (11) is necessary. Assume that (11) does not hold and let there exist $y \geq y_0$ such that

$$\int_Y^y \left[\frac{1}{a(\zeta)}\left[\int_\zeta^\infty c(\eta)\,d\eta\right]\right]^{1/\mu} d\zeta \leq \frac{(1-b_0)(g(1))^{-1/\mu}}{5},$$

where $\kappa > 0$ is a constant. We set

$$S = \left\{u \in C\bigl([y_0,\infty),\mathbb{R}\bigr) : u(y) = \frac{1-b_0}{5}, y \in [y_0, Y]\frac{1-b_0}{5} \leq u(y) \leq 1 \text{ for } y \geq Y\right\}.$$

We define the operator $\Omega : S \to C([y_0,\infty),\mathbb{R})$ by

$$(\Omega u)(y) = \begin{cases} \frac{1-b_0}{5}, & y \in [y_0, Y], \\ -b(y)u(\vartheta(y)) + \frac{1-b_0}{5} + \int_T^y [\frac{1}{a(\eta)}[\int_\eta^\infty c(\zeta)g(u(\varsigma(\zeta)))\,d\zeta]]^{1/\mu}\,d\eta, & y \geq T. \end{cases}$$

For every $u \in S$ and $y \geq Y$, $(\Omega u)(y) \geq \frac{1-b_0}{5}$ and

$$\begin{aligned}(\Omega u)(y) &\leq b_0 + \frac{1-b_0}{5} + \bigl(g(1)\bigr)^{1/\mu} \int_Y^y \left[\frac{1}{a(\eta)}\left[\int_\eta^\infty c(\zeta)\,d\zeta\right]\right]^{1/\mu} d\eta \\ &\leq b_0 + \frac{1-b_0}{5} + \frac{1-b_0}{5} = \frac{3b_0+2}{5} < 1,\end{aligned}$$

which implies that $\Omega u \in S$. The remaining proof follows from Theorem 3.1. This completes the proof. □

Example 3.6 Consider the differential equation

$$\Bigl(\Bigl(\bigl(u(y) - e^{-y}u\bigl(\vartheta(y)\bigr)\bigr)'\Bigr)^{1/5}\Bigr)' + (y+1)\bigl(u(y-2)\bigr)^{\frac{7}{3}} = 0, \quad y \geq 0. \tag{13}$$

Here $\mu = 1/5$, $a(y) = 1$, $\varsigma(y) = y - 2$, $g(v) = v^{\frac{7}{3}}$. For $\mu_1 = 4/3$, we have $g(v)/v^{\mu_1} = v$, which is an increasing function. To check (11) we have

$$\int_2^\infty \left[\int_\zeta^\infty (\eta+1)\,d\eta\right]^5 d\zeta = \infty.$$

So, all conditions of Theorem 3.5 hold, and therefore each solution of (13) oscillates or converges to zero.

4 Conclusion

It is worth noting that we have established the necessary and sufficient conditions when $-1 < b(y) \leq 0$. These conditions do not hold in all ranges of $b(y)$.

Remark 2 Theorems 3.1–3.5 also hold for the following equation:

$$\Bigl(a(y)\Bigl(\bigl(u(y) + b(y)u\bigl(\vartheta(y)\bigr)\bigr)'\Bigr)^\mu\Bigr)' + \sum_{i=1}^m c_j(y)g_j\bigl(u\bigl(\varsigma_j(y)\bigr)\bigr) = 0,$$

where $b, a, c_j, g_j, \varsigma_j$ $(j = 1, 2, \ldots, m)$ satisfy assumptions (A1)–(A5). In order to extend Theorems 3.1–3.5, we can find an index i so that c_j, g_j, ς_j satisfies (6) and (11).

Example 4.1 Consider the neutral differential equation

$$\left(e^{-y}\left(\left(u(y)-e^{-y}u\left(\vartheta(y)\right)\right)'\right)^{3/5}\right)' + \frac{1}{y+1}\left(u(y-2)\right)^{1/3} + \frac{1}{y+2}\left(u(y-1)\right)^{1/5} = 0, \quad y \geq 0. \tag{14}$$

Here $\mu = 3/5$, $a(y) = e^{-y}$, $b(y) = -e^{-y}$, $\varsigma_1(y) = u-2$, $\varsigma_2(y) = u-1$, $A(y) = \int_0^y e^{5s/3}\,ds = \frac{3}{5}(e^{5y/3} - 1)$, $g_1(v) = v^{1/3}$ and $g_2(v) = v^{1/5}$. For $\mu_1 = 1/2$, we have decreasing functions $g_1(v)/v^{\mu_1} = v^{-1/6}$ and $g_2(v)/v^{\mu_1} = v^{-3/10}$. Now,

$$\begin{aligned}
&\int_0^\infty \sum_{i=1}^m c_j(\eta) g_j\left(\kappa^{1/\mu} A\left(\varsigma_j(\eta)\right)\right) d\eta \\
&\quad \geq \int_0^\infty g_1(\eta) f_1\left(\kappa^{1/\mu} A\left(\varsigma_1(\eta)\right)\right) d\eta \\
&\quad = \int_0^\infty \frac{1}{\eta+1}\left(\kappa^{5/3}\frac{3}{5}\left(e^{5(\eta-2)/3} - 1\right)\right)^{1/3} d\eta = \infty \quad \forall \kappa > 0.
\end{aligned}$$

So, all the conditions of Theorem 3.1 hold, and therefore every unbounded solution of (14) is oscillatory.

Example 4.2 Consider the differential equation

$$\left(\left(\left(u(y) - e^{-y}u\left(\vartheta(y)\right)\right)'\right)^{5/7}\right)' + t\left(u(y-2)\right)^{5/3} + (y+1)\left(u(y-1)\right)^3 = 0, \quad y \geq 0. \tag{15}$$

Here $\mu = 5/7$, $a(y) = 1$, $\varsigma_1(y) = y-2$, $\varsigma_2(y) = y-1$, $g_1(v) = v^{5/3}$ and $g_2(v) = v^3$. For $\mu_1 = 4/3$, we have decreasing functions $g_1(v)/v^{\mu_1} = v^{1/3}$ and $g_2(v)/v^{\mu_1} = v^{5/3}$. Clearly, all the conditions of Theorem 3.5 hold. Thus, each solution of (15) oscillates or $\lim_{y\to\infty} u(y) = 0$.

Remark 3 Examples 4.1 and 4.2 prove the feasibility and effectiveness of Remark 2.

5 Open problem

This work leads to some open problems:

1. Can we find necessary and sufficient conditions for the oscillation of solutions to second-order differential equation (1) for the other ranges of the neutral coefficient b?
2. Is it possible to generalize this work to fractional order?

Acknowledgements
The authors are thankful to the the editors and and the referees for their valuable suggestions and comments, which improved the content of this paper.

Funding
The authors received no direct funding for this work.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that they read and approved the final manuscript.

Authors' information
Not applicable.

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Publisher's Note

Received: 21 July 2020 Accepted: 26 November 2020 Published online: 07 December 2020

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