Research Article

Leszek Gasiński*, Ireneusz Krech, and Nikolaos S. Papageorgiou

Positive solutions for parametric $(p(z), q(z))$-equations

https://doi.org/10.1515/math-2020-0074
received May 22, 2020; accepted August 17, 2020

Abstract: We consider a parametric elliptic equation driven by the anisotropic (p, q)-Laplacian. The reaction is superlinear. We prove a "bifurcation-type" theorem describing the change in the set of positive solutions as the parameter λ moves in $\mathbb{R}_+ = (0, +\infty)$.

Keywords: anisotropic regularity, anisotropic maximum principle, positive solutions, minimal positive solution, superlinear reaction

MSC 2020: 35J20, 35J70

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We study the following parametric anisotropic (p, q)-equation:

$$\begin{cases} -\Delta_{p(z)} u(z) - \Delta_{q(z)} u(z) = \lambda f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \lambda > 0. \end{cases} \tag{P_λ}$$

In this problem, the exponents p and q are Lipschitz continuous on $\overline{\Omega}$, that is, $p, q \in C^{0,1}(\overline{\Omega})$ and $1 < q_- = \min_{\overline{\Omega}} q \leqslant q_+ = \max_{\overline{\Omega}} q < p_- = \min_{\overline{\Omega}} p \leqslant p_+ = \max_{\overline{\Omega}} p$.

By $\Delta_{p(z)}$ (respectively $\Delta_{q(z)}$) we denote the $p(z)$-Laplacian (respectively the $q(z)$-Laplacian) defined by

$$\Delta_{p(z)} u = \operatorname{div}(|Du|^{p(z)-2} Du) \quad \forall u \in W_0^{1,p(z)}(\Omega)$$

$$(\text{respectively } \Delta_{q(z)} u = \operatorname{div}(|Du|^{q(z)-2} Du) \quad \forall u \in W_0^{1,q(z)}(\Omega)).$$

In the reaction (right hand side of (P_λ)), $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z, x)$ is continuous), which is $(p_+ - 1)$-superlinear in the x-variable, but need not satisfy the Ambrosetti-Rabinowitz condition which is common in problems with superlinear reactions. Also, $\lambda > 0$ is a parameter. We are looking for positive solutions of (P_λ). More precisely, our aim is to determine the precise dependence on the parameter $\lambda > 0$ of the set of positive solutions. We prove a bifurcation-type result, which establishes the existence of a critical parameter value $\lambda^* > 0$ such that

- for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least two positive solutions;
- for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution;
- for all $\lambda > \lambda^*$ there are no positive solutions for problem (P_λ).

*** Corresponding author: Leszek Gasiński,** Department of Mathematics, Pedagogical University of Cracow, Podchorazych 2, 30-084 Cracow, Poland, e-mail: leszek.gasinski@up.krakow.pl
Ireneusz Krech: Department of Mathematics, Pedagogical University of Cracow, Podchorazych 2, 30-084 Cracow, Poland
Nikolaos S. Papageorgiou: Department of Mathematics, National Technical University, Zografou Campus, Athens, 15780, Greece, e-mail: npapg@math.ntua.gr

Our work here extends those of Gasiński-Papageorgiou [1,2], who studied parametric equations driven by the isotropic p-Laplacian with a $(p-1)$-superlinear reaction. Nonlinear, nonparametric superlinear equations were also considered by Mugnai-Papageorgiou [3], Papageorgiou-Rădulescu [4], Papageorgiou-Scapellato [5] (isotropic problems) and Gasiński-Papageorgiou [6], Papageorgiou-Rădulescu-Repovš [7], Papageorgiou-Vetro [8] (anisotropic problems). They prove multiplicity results, producing also nodal (that is, sign changing) solutions. Also, we mention the relevant studies of Bahrouni-Rădulescu-Repovš [9] (existence of infinitely many solutions for anisotropic Dirichlet problems), Papageorgiou-Vetro-Vetro [10] (produce a continuous part of the spectrum for the Robin (p, q)-Laplacian), Vetro [11] (dealing with the asymptotic properties of the solutions of nonhomogeneous parametric isotropic equations), Vetro [12] (existence of a solution of an anisotropic Dirichlet problem), Vetro-Vetro [13] (a three-solution theorem for (p, q)-equations) and Vetro [14] (an infinity of solutions for isotropic (p, q)-equations).

Equations with variable exponents arise in many physical models. We refer to the book of Růžička [15] for such meaningful examples. The analysis of such problems requires the use of Lebesgue and Sobolev spaces with variable exponents. A comprehensive presentation of such spaces can be found in the book of Diening-Harjulehto-Hästö-Růžička [16] (see also the survey paper of Harjulehto-Hästö-Lê-Nuortio [17]). Various parametric boundary value problems with variable exponents can be found in the book of Rădulescu-Repovš [18]. Finally, we mention that we encounter (p, q)-equations (both isotropic and anisotropic), in many problems of mathematical physics. We refer to the studies of Bahrouni-Rădulescu-Repovš [19] (transonic flow problems), Benci-D'Avenia-Fortunato-Pisani [20] (quantum physics), Cherfils-Il'yasov [21] (reaction-diffusion systems) and Zhikov [22] (elasticity theory). We also mention the two informative survey papers by Marano-Mosconi [23] (isotropic problems) and Rădulescu [24] (isotropic and anisotropic problems).

2 Mathematical background – hypotheses

Let $M(\Omega)$ be the space of measurable functions $u : \Omega \to \mathbb{R}$. We identify two such functions that differ only on a set of zero Lebesgue measures. Also, let

$$E_1 = \{r \in C(\overline{\Omega}) : 1 < r_- = \min_{\overline{\Omega}} r\}.$$

In the sequel given $r \in C(\overline{\Omega})$, we define

$$r_- = \min_{\overline{\Omega}} q \quad \text{and} \quad r_+ = \max_{\overline{\Omega}} q.$$

Given $r \in E_1$, the variable exponent Lebesgue space $L^{r(z)}(\Omega)$ is defined as follows:

$$L^{r(z)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u|^{r(z)} \mathrm{d}z < +\infty \right\}.$$

This space is equipped with the so-called "Luxemburg norm" defined by

$$\|u\|_{r(z)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|u|}{\lambda} \right)^{r(z)} \mathrm{d}z \leqslant 1 \right\}.$$

Furnished with this norm, the space $L^{r(z)}(\Omega)$ becomes a separable, reflexive (in fact, uniformly convex) Banach space. Let $r' \in E_1$ be defined by $\frac{1}{r(z)} + \frac{1}{r'(z)} = 1$. We know that $L^{r(z)}(\Omega)^* = L^{r'(z)}(\Omega)$ and we have the following Hölder-type inequality:

$$\left| \int_\Omega uh \mathrm{d}z \right| \leqslant \left(\frac{1}{r_-} + \frac{1}{r'_-} \right) \|u\|_{r(z)} \|h\|_{r'(z)} \quad \forall u \in L^{r(z)}, h \in L^{r'(z)}(\Omega).$$

If $r_1, r_2 \in E_1$ and $r_1(z) \leqslant r_2(z)$ for all $z \in \overline{\Omega}$, then the embedding $L^{r_2(z)}(\Omega) \subseteq L^{r_1(z)}(\Omega)$ is continuous.

Using the variable exponent Lebesgue spaces, we can define in the usual way the variable exponent Sobolev spaces. So, if $r \in E_1$, then we define

$$W^{1,r(z)}(\Omega) = \{u \in L^{r(z)}(\Omega) : |Du| \in L^{r(z)}(\Omega)\}$$

(where the gradient Du is understood in the weak sense). This space is equipped with the following norm:

$$\|u\|_{1,r(z)} = \|u\|_{r(z)} + \||Du|\|_{r(z)}.$$

In the sequel for notational simplicity, we write $\|Du\|_{r(z)} = \||Du|\|_{r(z)}$. Suppose that $r \in E_1$ is Lipschitz continuous (that is, $r \in E_1 \cap C^{0,1}(\overline{\Omega})$). Then we define

$$W_0^{1,r(z)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,r(z)}}.$$

Both $W^{1,r(z)}(\Omega)$ and $W_0^{1,r(z)}(\Omega)$ are separable, reflexive (in fact uniformly convex) Banach spaces.

For the space $W_0^{1,r(z)}(\Omega)$, the Poincaré inequality holds, namely

$$\|u\|_{r(z)} \leqslant c_0 \|Du\|_{r(z)} \quad \forall u \in W_0^{1,r(z)}(\Omega),$$

for some $c_0 > 0$. So, on $W_0^{1,r(z)}(\Omega)$ (recall that $r \in E_1 \cap C^{0,1}(\overline{\Omega})$), we can consider the following equivalent norm:

$$\|u\|_{1,r(z)} = \|Du\|_{r(z)} \quad \forall u \in W_0^{1,r(z)}(\Omega).$$

For $r \in E_1$, the critical Sobolev exponent corresponding to r is defined by

$$r^*(z) = \begin{cases} \frac{Nr(z)}{N - r(z)} & \text{if } r(z) < N, \\ +\infty & \text{if } N \leqslant r(z). \end{cases}$$

Suppose that $r \in E_1 \cap C^{0,1}(\overline{\Omega})$, $p \in E_1$, $p_+ < N$ and $1 < p(z) \leqslant r^*(z)$ (respectively $1 < p(z) < r^*(z)$) for all $z \in \overline{\Omega}$. We have

$$W_0^{1,r(z)}(\Omega) \subseteq L^{p(z)}(\Omega) \text{ continuously}$$

(respectively: compactly).

Useful in the analysis of these variable exponent spaces is the following modular function:

$$\varrho_r(u) = \int_\Omega |u|^{r(z)} \mathrm{d}z \quad \forall u \in L^{r(z)}(\Omega),$$

with $r \in E_1$. We write $\varrho_r(Du) = \varrho_r(|Du|)$.

There is a close relation between this modular function and the norm. We assume $r \in E_1$.

Proposition 2.1.

(*a*) $\|u\|_{r(z)} = \lambda \Leftrightarrow \varrho_r\left(\frac{u}{\lambda}\right) = 1$ *for all* $u \in L^{r(z)}(\Omega)$, $u \neq 0$.

(*b*) $\|u\|_{r(z)} < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow \varrho_r(u) < 1$ (*resp.* $= 1, > 1$).

(*c*) $\|u\|_{r(z)} < 1 \Rightarrow \|u\|_{r(z)}^{r_+} \leqslant \varrho_r(u) \leqslant \|u\|_{r(z)}^{r_-}$.

(*d*) $\|u\|_{r(z)} > 1 \Rightarrow \|u\|_{r(z)}^{r_-} \leqslant \varrho_r(u) \leqslant \|u\|_{r(z)}^{r_+}$.

(*e*) $\|u_n\|_{r(z)} \to 0 \Leftrightarrow \varrho_r(u_n) \to 0$.

(*f*) $\|u_n\|_{r(z)} \to +\infty \Leftrightarrow \varrho_r(u_n) \to +\infty$.

More details can be found in the book of Diening-Harjulehto-Hästö-Růžička [16].

Consider the map $A_{r(z)} : W_0^{1,r(z)}(\Omega) \to W_0^{1,r(z)}(\Omega)^* = W^{-1,r'(z)}(\Omega)$ defined by

$$\langle A_{r(z)}(u), h\rangle = \int_\Omega |Du|^{r(z)-2}(Du, Dh)_{\mathbb{R}^N} \mathrm{d}z \quad \forall u, h \in W_0^{1,p}(\Omega).$$

This map has the following properties (see Gasiński-Papageorgiou [6, Proposition 2.5] and Rădulescu-Repovš [18, p. 40]).

Proposition 2.2. *The map* $A_{r(z)} : W_0^{1,r(z)}(\Omega) \to W_0^{1,r(z)}(\Omega)^*$ *is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type* $(S)_+$, *that is, "*$u_n \xrightarrow{w} u$ *in* $W_0^{1,r(z)}(\Omega)$ *and* $\limsup_{n\to+\infty}\langle A_{r(z)}(u_n), u_n - u\rangle \leqslant 0$, *imply that* $u_n \to u$ *in* $W_0^{1,r(z)}(\Omega)$*."*

In addition to the variable exponent spaces, we will also use the Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

This is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \ \text{for all} \ z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u > 0, \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} < 0\right\},$$

with n being the outward unit normal on $\partial\Omega$.

Given $u, v \in W^{1,r(z)}(\Omega)$ with $u \leqslant v$, we define

$$[u, v] = \{h \in W_0^{1,r(z)}(\Omega) : u(z) \leqslant h(z) \leqslant r(z) \ \text{for a.a.} \ z \in \Omega\},$$

$$[u) = \{h \in W_0^{1,r(z)}(\Omega) : u(z) \leqslant h(z) \ \text{for a.a.} \ z \in \Omega\}.$$

If $h_1, h_2 : \Omega \to \mathbb{R}$ are measurable functions, then we write $h_1 \prec h_2$, if for every compact set $K \subseteq \Omega$, we have $0 < c_K \leqslant h_2(z) - h_1(z)$ for almost all $z \in K$. Evidently, if $h_1, h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$ for all $z \in \Omega$, then $h_1 \prec h_2$.

A set $S \subseteq W_0^{1,p(z)}(\Omega)$ is said to be "downward directed," if for $u_1, u_2 \in S$, we can find $u \in S$ such that $u \leqslant u_1$, $u \leqslant u_2$.

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and by $\|\cdot\|_*$ the norm of $W_0^{1,p(z)}(\Omega)^*$.

In the sequel for notational economy, by $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p(z)}(\Omega)$. Recall that

$$\|u\| = \|Du\|_{p(z)} \quad \forall u \in W^{1,p(z)}(\Omega).$$

If X is a Banach space and $\varphi \in C^1(X)$, then we set

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}$$

(the critical set of φ). We say that φ satisfies the "Cerami condition," if the following property holds:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|_X)\varphi'(u_n) \to 0 \quad \text{in } X^* \quad \text{as } n \to +\infty,$$

admits a strongly convergent subsequence."

Now we introduce the hypotheses on the data problem (P_λ).

<u>H_0:</u> $p, q \in E_1 \cap C^{0,1}(\overline{\Omega})$, $q_+ < p_-$.

<u>H_1:</u> $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $f(z, x) \leqslant a(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)$ and $p_+ < r < p^*(z)$ for all $z \in \overline{\Omega}$;

(ii) if $F(z, x) = \int_0^x f(z, s)\,ds$, then

$$\lim_{x\to+\infty} \frac{F(z, x)}{x^{p_+}} = +\infty \ \text{uniformly for a.a.} \ z \in \Omega;$$

(iii) if $\sigma(z,x) = f(z,x)x - p_+F(z,x)$, then there exists $\eta \in L^1(\Omega)$ such that

$$\sigma(z,x) \leqslant \sigma(z,y) + \eta(z) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant y;$$

(iv) for every $s > 0$, there exists $m_s > 0$ such that

$$f(z,x) \geqslant m_s > 0 \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant s,$$

and

$$\lim_{x\to 0^+} \frac{f(z,x)}{x^{q_- -1}} = +\infty \text{ uniformly for a.a. } z \in \Omega;$$

(v) for every $\varrho > 0$, there exists $\hat{\xi}_\varrho > 0$ such that for a.a. $z \in \Omega$, the function $x \mapsto f(z,x) + \hat{\xi}_\varrho x^{p(z)-1}$ is nondecreasing on $[0, \varrho]$.

Remark 2.3. Since we want to find positive solutions and the aforementioned hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that

$$f(z,x) = 0 \text{ for a.a. } z \in \Omega, \text{ all } x \leqslant 0. \tag{2.1}$$

Hypotheses $H_1(ii)$, (iii) imply that $f(z,\cdot)$ is $(p_+ - 1)$-superlinear. Usually in the literature, such problems are treated using the well-known Ambrosetti-Rabinowitz condition (see Ambrosetti-Rabinowitz [25]). Here instead we use the less restrictive condition $H_1(iii)$, which is an extension of a condition used by Li-Yang [26]. This quasimonotonicity condition on the function $\sigma(z,\cdot)$ is equivalent to saying that there exists $M > 0$ such that for a.a. $z \in \Omega$, the quotient function $x \mapsto \frac{f(z,x)}{x^{p_+-1}}$ is nondecreasing on $[M, +\infty)$. This superlinearity condition incorporates in our framework superlinear nonlinearities with "slower" growth near $+\infty$. For example, consider the following function:

$$f(z,x) = \begin{cases} x^{\tau(z)-1} & \text{if } 0 \leqslant x \leqslant 1, \\ x^{p_+-1}\ln x + x^{\mu(z)-1} & \text{if } 1 < x \end{cases}$$

(see (2.1)), with $\tau, \mu \in E_1$ and $\tau_+ < q_-$, $\mu(z) \leqslant p(z)$ for all $z \in \overline{\Omega}$. This function satisfies hypotheses H_1, but fails to satisfy the Ambrosetti-Rabinowitz condition.

We introduce the following two sets:

$$\mathcal{L} = \{\lambda > 0 : \text{problem } (P_\lambda) \text{ admits a positive solution}\},$$

$$S_\lambda = \text{the set of positive solutions of } (P_\lambda).$$

Also, we set

$$\lambda^* = \sup \mathcal{L}.$$

3 Positive solutions

We start by showing that the set of admissible parameters $\mathcal{L}$ is nonempty. Also, we determine the regularity properties of the elements in S_λ.

Proposition 3.1. *If hypotheses H_0, $H_1(i)$, (iv) hold, then $\mathcal{L} \neq \emptyset$ and for every $\lambda > 0$, $S_\lambda \subseteq \operatorname{int} C_+$.*

Proof. We consider the following auxiliary Dirichlet problem:

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) = 1 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{3.1}$$

The operator $u \mapsto A_{p(z)}(u) + A_{q(z)}(u)$ from $W_0^{1,p(z)}(\Omega)$ into $W_0^{1,p(z)}(\Omega)^*$ is continuous, strictly monotone (hence maximal monotone too) (see Proposition 2.2) and coercive. So, it is surjective (see Gasiński-Papageorgiou [27, Corollary 3.2.31, p. 319]). Hence, we can find $\bar{u} \in W_0^{1,p(z)}(\Omega)$, $\bar{u} \geqslant 0$, $\bar{u} \neq 0$ such that

$$A_{p(z)}(\bar{u}) + A_{q(z)}(\bar{u}) = 1 \text{ in } W_0^{1,p(z)}(\Omega)^*.$$

The strict monotonicity of the operator implies that this solution is unique. So, $\bar{u}$ is the unique positive solution of (3.1). Theorem 4.1 of Fan-Zhao [28] implies that $\bar{u} \in L^\infty(\Omega)$. Then from Fukagai-Narukawa [29, Lemma 3.3] (see also Tan-Fang [30, Corollary 3.1] and Lieberman [31] for the corresponding isotropic regularity theory), we have that $\bar{u} \in C_0^{1,\alpha}(\overline{\Omega}) = C^{1,\alpha}(\overline{\Omega}) \cap C_0^1(\overline{\Omega})$ with $\alpha \in (0,1)$. Hence, $\bar{u} \in C_+ \setminus \{0\}$. From the anisotropic maximum principle (see Zhang [32]), we obtain that $\bar{u} \in \operatorname{int} C_+$.

Let $m = \|f(\cdot, \bar{u}(\cdot))\|_\infty$ (see hypothesis $H_1(i)$) and choose $\lambda_0 > 0$ such that $\lambda_0 m \leqslant 1$. We have

$$-\Delta_{p(z)}\bar{u} - \Delta_{q(z)}\bar{u} \geqslant \lambda f(z, \bar{u}) \text{ in } \Omega, \tag{3.2}$$

for all $\lambda \in (0, \lambda_0]$. We introduce the Carathéodory function $\hat{g}(z, x)$ defined by

$$\hat{g}(z,x) = \begin{cases} f(z, x^+) & \text{if } x \leqslant \bar{u}(z), \\ f(z, \bar{u}(z)) & \text{if } \bar{u}(z) < x. \end{cases} \tag{3.3}$$

We set

$$\hat{G}(z,x) = \int_0^x \hat{g}(z,s)\,\mathrm{d}s$$

and for all $\lambda \in (0, \lambda_0]$ consider the C^1-functional $\hat{\varphi}_\lambda : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,\mathrm{d}z + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,\mathrm{d}z - \int_\Omega \lambda \hat{G}(z,u)\,\mathrm{d}z \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

From (3.3) and Proposition 2.1, it is clear that $\hat{\varphi}_\lambda$ is coercive. Also, the anisotropic Sobolev embedding theorem implies that $\hat{\varphi}_\lambda$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_\lambda \in W_0^{1,p(z)}(\Omega)$ such that

$$\hat{\varphi}_\lambda(\hat{u}_\lambda) = \min_{u \in W_0^{1,p(z)}(\Omega)} \hat{\varphi}_\lambda(u). \tag{3.4}$$

Hypothesis $H_1(iv)$ implies that given any $\theta > 0$, we can find $\delta = \delta(\theta) \in (0,1)$ such that

$$F(z,x) \geqslant \frac{\theta}{q_-} x^{q_-} \text{ for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta. \tag{3.5}$$

Let $u \in \operatorname{int} C_+$. Since $\bar{u} \in \operatorname{int} C_+$, using Proposition 4.1.22 of Papageorgiou-Rădulescu-Repovš [33, p. 274], we can find $t \in (0,1)$ small such that

$$tu(z) \leqslant \min\{\bar{u}(z), \delta\} \text{ for all } z \in \overline{\Omega}. \tag{3.6}$$

From (3.3), (3.5) and (3.6) and since $t \in (0,1)$, we have

$$\hat{\varphi}_\lambda(tu) \leqslant \frac{t^{q_-}}{q_-}(\varrho_p(Du) + \varrho_q(Du) - \theta\|u\|_{q_-}^{q_-}).$$

Since $\theta > 0$ is arbitrary, choosing $\theta > 0$ big from the aforementioned inequality, we infer that

$$\hat{\varphi}_\lambda(tu) < 0,$$

so

$$\hat{\varphi}_\lambda(u_\lambda) < 0 = \hat{\varphi}_\lambda(0)$$

(see (3.4)) and thus $u_\lambda \neq 0$.

From (3.4), we have

$$\hat{\varphi}'_\lambda(u_\lambda) = 0,$$

so

$$\langle A_{p(z)}(u_\lambda), h\rangle + \langle A_{q(z)}(u_\lambda), h\rangle = \int_\Omega \lambda \hat{g}(z, u_\lambda) h \mathrm{d}z \quad \forall h \in W_0^{1,p(z)}(\Omega). \tag{3.7}$$

We test (3.7) with $h = -u_\lambda^- \in W^{1,p(z)}(\Omega)$ and obtain

$$\varrho_p(Du_\lambda^-) + \varrho_q(Du_\lambda^-) = 0,$$

so $u_\lambda \geqslant 0$, $u_\lambda \neq 0$ (see Proposition (2.1)).

Next in (3.7) we choose $h = (u_\lambda - \bar{u})^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned} &\langle A_{p(z)}(u_\lambda), (u_\lambda - \bar{u})^+\rangle + \langle A_{q(z)}(u_\lambda), (u_\lambda - \bar{u})^+\rangle \\ &= \int_\Omega \lambda f(z, \bar{u})(u_\lambda - \bar{u})^+ \mathrm{d}z \leqslant \langle A_{p(z)}(\bar{u}), (u_\lambda - \bar{u})^+\rangle + \langle A_{q(z)}(\bar{u}), (u_\lambda - \bar{u})^+\rangle \end{aligned}$$

(see (3.3) and (3.2)), so

$$u_\lambda \leqslant \bar{u}.$$

So, we have proved that

$$u_\lambda \in [0, \bar{u}], u_\lambda \neq 0. \tag{3.8}$$

From (3.8), (3.3) and (3.7), it follows that u_λ is a positive solution of (P_λ). As before using the anisotropic regularity theorem (see Fan-Zhao [28], Fukagai-Narukawa [29]) and the anisotropic maximum principle (see Zhang [32]), we obtain that $u_\lambda \in \operatorname{int} C_+$.

Therefore, we conclude that

$$(0, \lambda_0] \subseteq \mathcal{L} \neq \varnothing$$

and

$$S_\lambda \subseteq \operatorname{int} C_+ \quad \lambda > 0. \qquad \square$$

Next, we show that $\mathcal{L}$ is connected.

Proposition 3.2. *If hypotheses H_0, $H_1(i)$, (iv) hold, $\lambda \in \mathcal{L}$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ (see Proposition 3.1). We introduce the Carathéodory function $\tilde{g}$ defined by

$$\tilde{g}(z, x) = \begin{cases} f(z, x^+) & \text{if } x \leqslant u_\lambda(z), \\ f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.9}$$

We set

$$\tilde{G}(z, x) = \int_0^x \tilde{g}(z, s) \mathrm{d}s$$

and consider the C^1-functional $\tilde{\varphi}_\mu : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\varphi}_\mu(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} \mathrm{d}z + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} \mathrm{d}z - \int_\Omega \mu \tilde{G}(z, u) \mathrm{d}z \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

On account of (3.9) and Proposition 2.1, $\tilde{\varphi}_\mu$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\mu \in W_0^{1,p(z)}(\Omega)$ such that

$$\tilde{\varphi}_\mu(u_\mu) = \min_{u \in W_0^{1,p(z)}(\Omega)} \tilde{\varphi}_\mu(u) < 0 = \tilde{\varphi}_\mu(0)$$

(see the proof of Proposition 3.1), thus $u_\mu \neq 0$. We have

$$\tilde{\varphi}_\mu'(u_\mu) = 0$$

and from this as in the proof of Proposition 3.1, using (3.9), we obtain

$$u_\mu \in [0, u_\lambda], \quad u_\mu \neq 0,$$

so

$$u_\mu \in S_\mu \subseteq \operatorname{int} C_+$$

(see (3.9) and Proposition 3.1), hence $\mu \in \mathcal{L}$. □

A byproduct of the above proof is the following monotonicity property of the solution multifunction $\lambda \mapsto S_\lambda$.

Corollary 3.3. *If hypotheses H_0, $H_1(i)$, (iv) hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that $u_\mu \leqslant u_\lambda$.*

We can improve this monotonicity property, if we bring in the picture hypothesis $H_1(v)$.

Proposition 3.4. *If hypotheses H_0, $H_1(i)$, (iv), (v) hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that*

$$u_\lambda - u_\mu \in \operatorname{int} C_+.$$

Proof. From Corollary 3.3, we already know that $\mu \in \mathcal{L}$ and that there exists $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that $u_\mu \leqslant u_\lambda$. Let $\varrho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\varrho > 0$ be as postulated by hypothesis $H_1(v)$. We have

$$\begin{aligned} -\Delta_{p(z)} u_\mu - \Delta_{q(z)} u_\mu + \hat{\xi}_\varrho u_\mu^{p(z)-1} &= \mu f(z, u_\mu) + \hat{\xi}_\varrho u_\mu^{p(z)-1} = \lambda f(z, u_\mu) + \hat{\xi}_\varrho u_\mu^{p(z)-1} - (\lambda - \mu) f(z, u_\mu) \\ &\leqslant \lambda f(z, u_\lambda) + \hat{\xi}_\varrho u_\lambda^{p(z)-1} (\lambda - \mu) f(z, u_\mu) \leqslant -\Delta_{p(z)} u_\lambda - \Delta_{q(z)} u_\lambda + \hat{\xi}_\varrho u_\lambda^{p(z)-1} \end{aligned} \tag{3.10}$$

(see hypothesis $H_1(v)$). Since $u_\mu \in \operatorname{int} C_+$, on account of hypothesis $H_1(iv)$, we have that

$$0 \prec (\lambda - \mu) f(\cdot, u_\mu(\cdot)).$$

Then from (3.10) and Proposition 2.4 of Papageorgiou-Rădulescu-Repovš [7], we conclude that $u_\lambda - u_\mu \in \operatorname{int} C_+$. □

Next for every $\lambda \in \mathcal{L}$, we will produce a smallest (minimal) positive solution for problem (P_λ). To this end, we need some preparation.

Hypotheses $H_1(i)$, (iv) imply that given $\beta > 0$, we can find $c_1 = c_1(\beta) > 0$ such that

$$f(z, x) \geqslant \beta x^{q_- - 1} - c_1 x^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{3.11}$$

Motivated from this unilateral growth estimate for $f(z, \cdot)$, we consider the following auxiliary Dirichlet problem:

$$\begin{cases} -\Delta_{p(z)} u(z) - \Delta_{q(z)} u(z) = \lambda(\beta u(z)^{q_- - 1} - c_1 u(z)^{r-1}) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \lambda > 0. \end{cases} \tag{Q_λ}$$

Proposition 3.5. *For every $\lambda > 0$, we can choose $\beta = \beta(\lambda) > 0$ big such that (Q_λ) has a unique positive solution $\tilde{u}_\lambda \in \operatorname{int} C_+$.*

Proof. First we show the existence of a positive solution. To this end, we consider the C^1-functional $\sigma_\lambda : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\lambda(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz + \frac{\lambda c_1}{r}\|u^+\|_r^r - \frac{\lambda\beta}{q_-}\|u^+\|_{q_-}^{q_-} \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

Since $q_- \leqslant q(z) < p(z) \leqslant p_+ < r$ for all $z \in \overline{\Omega}$ (see hypothesis H_0), we see that σ_λ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W_0^{1,p(z)}(\Omega)$ such that

$$\sigma_\lambda(\tilde{u}_\lambda) = \min_{u \in W_0^{1,p(z)}(\Omega)} \sigma_\lambda(u). \tag{3.12}$$

Consider $u \in C_+$ with $\|u\|_\infty \leqslant 1$. For $t \in (0,1)$, we have

$$\begin{aligned}\sigma_\lambda(tu) &\leqslant \frac{t^{p_-}}{p_-}\varrho_p(Du) + \frac{t^{q_-}}{q_-}(\varrho_q(Du) - \lambda\beta\|u\|_{q_-}^{q_-}) + \frac{\lambda c_1}{r}t^r\|u\|_r^r \\ &\leqslant \frac{t^{q_-}}{q_-}(\varrho_p(Du) + \varrho_q(Du) - \lambda\beta\|u\|_{q_-}^{q_-}) + \frac{\lambda c_1}{r}t^r\|u\|_r^r.\end{aligned}$$

Recall that $\beta > 0$ is arbitrary. So, we choose $\beta_\lambda > \frac{\varrho_p(Du)+\varrho_q(Du)}{\lambda \| u \|_{q_-}^{q_-}}$ and obtain

$$\sigma_\lambda(tu) \leqslant \lambda c_2 t^r - c_3 t^{q_-},$$

for some $c_2, c_3 > 0$. Since $q_- < r$, choosing $t \in (0,1)$ small, we have

$$\sigma_\lambda(tu) < 0,$$

so

$$\sigma_\lambda(\tilde{u}_\lambda) < 0 = \sigma_\lambda(0)$$

(see (3.12)), hence $\tilde{u}_\lambda \neq 0$.

From (3.12), we have

$$\sigma_\lambda'(\tilde{u}_\lambda) = 0,$$

so

$$\langle A_{p(z)}(\tilde{u}_\lambda), h\rangle + \langle A_{q(z)}(\tilde{u}_\lambda), h\rangle = \lambda\beta_\lambda \int_\Omega (\tilde{u}_\lambda^+)^{q_- - 1} h\,dz - \lambda c_1 \int_\Omega (\tilde{u}_\lambda^+)^{r-1} h\,dz \quad \forall h \in W_0^{1,p(z)}(\Omega). \tag{3.13}$$

In (3.13), we choose $h = -\tilde{u}^- \in W_0^{1,p(z)}(\Omega)$ and obtain

$$\varrho_p(D\tilde{u}_\lambda^-) + \varrho_q(D\tilde{u}_\lambda^-) = 0,$$

so $\tilde{u}_\lambda \geqslant 0$, $\tilde{u}_\lambda \neq 0$ (see Proposition 2.1).

Then from (3.13) we infer that $\tilde{u}_\lambda$ is a positive solution of (Q_λ). Moreover, as before the anisotropic regularity theory and the anisotropic maximum principle imply

$$\tilde{u}_\lambda \in \operatorname{int} C_+. \tag{3.14}$$

Let $\tilde{v}_\lambda \in W_0^{1,p(z)}(\Omega)$ be another positive solution of (Q_λ). Again we have

$$\tilde{v}_\lambda \in \operatorname{int} C_+. \tag{3.15}$$

We consider the integral functional $j : L^1(\Omega) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega \frac{q_-}{p(z)}|Du^{\frac{1}{q_-}}|^{p(z)}\,dz + \int_\Omega \frac{q_-}{q(z)}|Du^{\frac{1}{q_-}}|^{q(z)}\,dz & \text{if } u \geqslant 0,\ u^{\frac{1}{q_-}} \in W_0^{1,p(z)}(\Omega), \\ +\infty, & \text{otherwise.}\end{cases}$$

From Theorem 2.2 of Takáč-Giacomoni [34], we have that j is convex. Let

$$\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < +\infty\}$$

(the effective domain of j). From (3.14), (3.15) and Proposition 4.1.22 of Papageorgiou-Rădulescu-Repovš [33, p. 274], we have

$$\frac{\tilde{u}_\lambda}{\tilde{v}_\lambda} \in L^\infty(\Omega), \quad \frac{\tilde{v}_\lambda}{\tilde{u}_\lambda} \in L^\infty(\Omega).$$

Let $h \in C_0^1(\overline{\Omega})$ with $|h|^{\frac{1}{q_-}} \in W_0^{1,p(z)}(\Omega)$. For $t \in (0,1)$ small, we have

$$\tilde{u}_\lambda^{q_-} + th \in \operatorname{dom} j \quad \text{and} \quad \tilde{v}_\lambda^{q_-} + th \in \operatorname{dom} j.$$

Choose $h = \tilde{u}_\lambda^{q_-} - \tilde{v}_\lambda^{q_-}$. Evidently,

$$h \in C_0^1(\overline{\Omega}) \quad \text{and} \quad |h| \leqslant \tilde{u}_\lambda^{q_-} + \tilde{v}_\lambda^{q_-}.$$

We have

$$|h|^{\frac{1}{q_-}} \leqslant \tilde{u}_\lambda + \tilde{v}_\lambda,$$

so $|h|^{\frac{1}{q_-}} \in W_0^{1,p(z)}(\Omega)$.

Then on account of the convexity of j, it is Gâteaux differentiable at $\tilde{u}_\lambda^{q_-}$ and at $\tilde{v}_\lambda^{q_-}$ in the direction $h = \tilde{u}_\lambda^{q_-} - \tilde{v}_\lambda^{q_-}$. Moreover, we have (see also Takáč-Giacomoni [34])

$$j'(\tilde{u}_\lambda^{q_-})(h) = \int_\Omega \frac{-\Delta_{p(z)}\tilde{u}_\lambda - \Delta_{q(z)}\tilde{u}_\lambda}{\tilde{u}_\lambda^{q_- - 1}} h \, dz,$$

$$j'(\tilde{v}_\lambda^{q_-})(h) = \int_\Omega \frac{-\Delta_{p(z)}\tilde{v}_\lambda - \Delta_{q(z)}\tilde{v}_\lambda}{\tilde{v}_\lambda^{q_- - 1}} h \, dz.$$

The convexity of j implies the monotonicity of j'. Hence,

$$0 \leqslant \lambda c_1 \int_\Omega (\tilde{u}_\lambda^{r-q} - \tilde{v}_\lambda^{r-q_-})(\tilde{v}_\lambda^{q_-} - \tilde{u}_\lambda^{q_-}) \, dz \leqslant 0$$

(recall that $q_- < r$), so

$$\tilde{u}_\lambda = \tilde{v}_\lambda,$$

thus $\tilde{u}_\lambda \in \operatorname{int} C_+$ is the unique positive solution of (Q_λ). □

Using $\tilde{u}_\lambda \in \operatorname{int} C_+$ from Proposition 3.5, we can have a lower bound for the elements of S_λ.

Proposition 3.6. *If hypotheses H_0, $H_1(i)$, (iv), (v) hold and $\lambda \in \mathcal{L}$, then $\tilde{u}_\lambda \leqslant u$ for all $u \in S_\lambda$.*

Proof. Let $u \in S_\lambda$. We introduce the Carathéodory function $k(z,x)$ defined by

$$k(z,x) = \begin{cases} \beta(x^+)^{q_- - 1} - c_1(x^+)^{r-1} & \text{if } x \leqslant u(z), \\ \beta u(z)^{q_- - 1} - c_1 u(z)^{r-1} & \text{if } u(z) < x. \end{cases} \tag{3.16}$$

We set

$$K(z,x) = \int_0^x k(z,s) \, ds$$

and consider the C^1-functional $\gamma_\lambda : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_\lambda(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz - \int_\Omega \lambda K(z,u)\,dz \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

From Proposition 2.1 and (3.16) it is clear that γ_λ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\bar{u}_\lambda \in W_0^{1,p(z)}(\Omega)$ such that

$$\gamma_\lambda(\bar{u}_\lambda) = \min_{u \in W_0^{1,p(z)}(\Omega)} \gamma_\lambda(u). \tag{3.17}$$

As before (see the proof of Proposition 3.5), we have

$$\gamma_\lambda(\bar{u}_\lambda) < 0 = \gamma_\lambda(0),$$

so $\bar{u}_\lambda \neq 0$.

From (3.17), we have

$$\gamma_\lambda'(\bar{u}_\lambda) = 0,$$

so

$$\langle A_{p(z)}(\bar{u}_\lambda), h\rangle + \langle A_{q(z)}(\bar{u}_\lambda), h\rangle = \lambda \int_\Omega k(z, \bar{u}_\lambda)h\,dz \quad \forall h \in W_0^{1,p(z)}(\Omega). \tag{3.18}$$

We test (3.18) with $h = -\bar{u}_\lambda^- \in W_0^{1,p(z)}(\Omega)$ and obtain

$$\varrho_p(D\bar{u}_\lambda^-) + \varrho_q(D\bar{u}_\lambda^-) = 0$$

(see (3.16)), so $\bar{u}_\lambda \geqslant 0$, $\bar{u}_\lambda \neq 0$.

Next in (3.18) we choose $h = (\bar{u}_\lambda - u)^+ \in W_0^{1,p(z)}(\Omega)$. Then

$$\begin{aligned}\langle A_{p(z)}(\bar{u}_\lambda), (\bar{u}_\lambda - u)^+\rangle + \langle A_{q(z)}(\bar{u}_\lambda), (\bar{u}_\lambda - u)^+\rangle &= \int_\Omega \lambda(\beta u^{q_- -1} - c_1 u^{r-1})(\bar{u}_\lambda - u)^+ dz \leqslant \int_\Omega \lambda f(z,u)(\bar{u}_\lambda - u)^+ dz \\ &= \langle A_{p(z)}(u), (\bar{u}_\lambda - u)^+\rangle + \langle A_{q(z)}(u), (\bar{u}_\lambda - u)^+\rangle\end{aligned}$$

(see (3.16), (3.11) and use the fact that $u \in S_\lambda$), so $\bar{u}_\lambda \leqslant u$ (see Proposition 2.2).

So, we have proved that

$$\bar{u}_\lambda \in [0,u], \quad \bar{u}_\lambda \neq 0. \tag{3.19}$$

From (3.19), (3.16) and (3.18), it follows that $\bar{u}_\lambda$ is a positive solution of (Q_λ), hence $\bar{u}_\lambda = \tilde{u}_\lambda$ (see Proposition (3.6)). We conclude that $\tilde{u}_\lambda \leqslant u$ for all $u \in S_\lambda$. ☐

Now we are ready to produce the minimal positive solution of problem (P_λ), $\lambda \in \mathcal{L}$.

Proposition 3.7. *If hypotheses H_0, $H_1(i), (iv), (v)$ hold and $\lambda \in \mathcal{L}$, then problem (P_λ) admits a smallest positive solution $u_\lambda^* \in S_\lambda \subseteq \operatorname{int} C_+$ (that is, $u_\lambda^* \leqslant u$ for all $u \in S_\lambda$).*

Proof. From Papageorgiou-Rădulescu-Repovš [35] (proof of Proposition 7; see also Filippakis-Papageorgiou [36]), we know that S_λ is downward directed. So, by Lemma 3.10 of Hu-Papageorgiou [37, p. 178], we can find a decreasing sequence $\{u_n\}_{n\geqslant 1} \subseteq S_\lambda$ such that

$$\inf S_\lambda = \inf_{n\geqslant 1} u_n \tag{3.20}$$

and

$$\tilde{u}_\lambda \leqslant u_n \leqslant u_1 \quad \forall n \in \mathbb{N} \tag{3.21}$$

(see Proposition 3.6). We have

$$\langle A_{p(z)}(u_n), h\rangle + \langle A_{q(z)}(u_n), h\rangle = \int_\Omega \lambda f(z, u_n)h\,dz \quad \forall h \in W_0^{1,p(z)}(\Omega), n \in \mathbb{N}. \tag{3.22}$$

In (3.22), we use $h = u_n \in W_0^{1,p(z)}(\Omega)$. From (3.21), hypothesis $H_1(i)$ and Proposition 2.1, it follows that the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded.

So, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W_0^{1,p(z)}(\Omega) \quad \text{and} \quad u_n \to u_\lambda^* \text{ in } L^{p(z)}(\Omega). \tag{3.23}$$

We test (3.22) with $h = u_n - u_\lambda^* \in W_0^{1,p(z)}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.23). We obtain

$$\lim_{n\to+\infty} (\langle A_{p(z)}(u_n), u_n - u_\lambda^*\rangle + \langle A_{q(z)}(u_n), u_n - u_\lambda^*\rangle) = 0,$$

so

$$\limsup_{n\to+\infty}(\langle A_{p(z)}(u_n), u_n - u_\lambda^*\rangle + \langle A_{q(z)}(u_\lambda^*), u_n - u_\lambda^*\rangle) \leqslant 0$$

(since $A_{q(z)}$ is monotone), thus

$$\limsup_{n\to+\infty}\langle A_{p(z)}(u_n), u_n - u_\lambda^*\rangle \leqslant 0$$

(see (3.23)) and hence

$$u_n \to u_\lambda^* \text{ in } W_0^{1,p(z)}(\Omega) \tag{3.24}$$

(see Proposition 2.2).

Then passing to the limit as $n \to +\infty$ in (3.22) and using (3.24) and (3.21), we obtain

$$\langle A_{p(z)}(u_\lambda^*), h\rangle + \langle A_{q(z)}(u_\lambda^*), h\rangle = \int_\Omega \lambda f(z, u_\lambda^*) h dz \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so

$$\tilde{u}_\lambda \leqslant u_\lambda^*$$

and hence

$$u_\lambda^* \in S_\lambda \subseteq \operatorname{int} C_+, \quad u_\lambda^* = \inf S_\lambda. \qquad \square$$

We consider the map $\lambda \mapsto u_\lambda^*$ from $\mathcal{L}$ into $C_0^1(\overline{\Omega})$.

Proposition 3.8. *If hypotheses H_0, $H_1(i)$, (iv), (v) hold, then the map $\lambda \mapsto u_\lambda^*$ from $\mathcal{L}$ into $C_0^1(\overline{\Omega})$ is*
(*a*) *strictly increasing (that is, if $0 < \mu < \lambda \in \mathcal{L}$, then $u_\lambda^* - u_\mu^* \in \operatorname{int} C_+$);*
(*b*) *left continuous.*

Proof. (a) Suppose that $0 < \mu < \lambda \in \mathcal{L}$. Let $u_\lambda^* \in S_\lambda \subseteq \operatorname{int} C_+$ be the minimal solution of problem (P_λ) (see Proposition 3.7). According to Proposition 3.4, we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that

$$u_\lambda^* - u_\mu \in \operatorname{int} C_+,$$

so

$$u_\lambda^* - u_\mu^* \in \operatorname{int} C_+$$

and hence the map $\lambda \mapsto u_\lambda^*$ is strictly increasing.

(b) Let $\lambda_n \to \lambda^-$ with $\lambda \in \mathcal{L}$. Let $u_n^* = u_{\lambda_n}^* \in \operatorname{int} C_+$ for all $n \in \mathbb{N}$. From part (a) and hypothesis $H_1(i)$, we see that the sequence $\{u_n^*\}_{n\geqslant 1} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded.

Then from the anisotropic regularity theory (see Fukagai-Narukawa [29] and Tan-Fang [30]), we can find $\alpha \in (0, 1)$ and $c_4 > 0$ such that

$$u_n^* \in C_0^{1,\alpha}(\overline{\Omega}), \|u_n^*\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant c_4 \quad \forall n \in \mathbb{N}.$$

Exploiting the compactness of the embedding $C_0^{1,\alpha}(\overline{\Omega}) \subseteq C_0^1(\overline{\Omega})$, we have

$$u_n^* \to \hat{u}_\lambda^* \text{ in } C_0^1(\overline{\Omega}). \tag{3.25}$$

Evidently $\hat{u}_\lambda^* \in S_\lambda$. If $\hat{u}_\lambda^* \neq u_\lambda^*$, then we can find $z_0 \in \Omega$ such that

$$u_\lambda^*(z_0) < \hat{u}_\lambda^*(z_0),$$

so

$$u_\lambda^*(z_0) < u_n^*(z_0) \quad \forall n \geqslant n_0$$

(see (3.25)). This contradicts part (a). So, the map $\lambda \mapsto u_\lambda^*$ is left continuous. □

So far, we only know that $\mathcal{L}$ is nonempty and connected. We do not know if it is bounded or not. The next proposition shows that $\mathcal{L}$ is bounded. In what follows, by $\varphi_\lambda : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ we denote the energy (Euler) functional of problem (P_λ) defined by

$$\varphi_\lambda(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz - \int_\Omega \lambda F(z,u)\,dz \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

Proposition 3.9. *If hypotheses H_0, H_1 hold, then $\lambda^* < +\infty$.*

Proof. We argue by contradiction. So, suppose that $\lambda^* = +\infty$ (that is, $\mathcal{L} = (0, +\infty)$). Let $\{\lambda_n\}_{n\geqslant 1} \subseteq \mathcal{L}$ be such that $\lambda_n \nearrow +\infty$. Then on account of Proposition 3.8 and hypothesis $H_1(ii)$, we can find a nondecreasing sequence $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$ for $n \in \mathbb{N}$ such that

$$\varphi_{\lambda_n}(u_n) \leqslant c_5 \quad \forall n \in \mathbb{N}, \tag{3.26}$$

for some $c_5 > 0$ and

$$\varphi'_{\lambda_n}(u_n) = 0 \quad \forall n \in \mathbb{N}. \tag{3.27}$$

From (3.27), we have

$$\langle A_{p(z)}(u_n), h\rangle + \langle A_{q(z)}(u_n), h\rangle = \lambda \int_\Omega f(z,u_n)h\,dz \quad \forall h \in W_0^{1,p(z)}(\Omega). \tag{3.28}$$

We test (3.28) with $h = u_n \in W_0^{1,p(z)}(\Omega)$. Then

$$-\varrho_p(Du_n) - \varrho_q(Du_n) + \lambda_n \int_\Omega f(z,u_n)u_n\,dz = 0 \quad \forall n \in \mathbb{N}. \tag{3.29}$$

Also from (3.26), we have

$$\int_\Omega \frac{1}{p(z)}|Du_n|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du_n|^{q(z)}\,dz - \lambda_n \int_\Omega F(z,u_n)\,dz \leqslant c_5 \quad \forall n \in \mathbb{N},$$

so

$$\frac{1}{p_+}(\varrho_p(Du_n) + \varrho_q(Du_n)) - \lambda_n \int_\Omega F(z,u_n)\,dz \leqslant c_5 \quad \forall n \in \mathbb{N},$$

thus

$$\varrho_p(Du_n) + \varrho_q(Du_n) - \lambda_n \int_\Omega p_+F(z,u_n)\,dz \leqslant p_+c_5 \quad \forall n \in \mathbb{N}. \tag{3.30}$$

Adding (3.29) and (3.30), we obtain

$$\lambda_n \int_\Omega \sigma(z, u_n)\mathrm{d}z \leqslant p_+ c_5 \quad \forall n \in \mathbb{N}$$

so

$$\int_\Omega \sigma(z, u_n)\mathrm{d}z \leqslant \frac{p_+ c_5}{\lambda_n} \quad \forall n \in \mathbb{N}. \tag{3.31}$$

Suppose that the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p(z)}(\Omega)$ is not bounded. We may assume that

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty. \tag{3.32}$$

We set $y_n = \frac{u_n}{\|u_n\|}$ for $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geqslant 0$ for all $n \in \mathbb{N}$. We may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p(z)}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^{p(z)}(\Omega), y \geqslant 0. \tag{3.33}$$

First suppose that $y \neq 0$. Let $\widehat{\Omega} = \{y > 0\}$. Then $|\widehat{\Omega}|_N > 0$ (see (3.33)) and $u_n(z) \to +\infty$ for almost all $z \in \widehat{\Omega}$. On account of hypothesis $H_1(ii)$, we have

$$\frac{F(z, u_n(z))}{\|u_n\|^{p_+}} = \frac{F(z, u_n(z))}{u_n(z)^{p^+}} y_n(z)^{p_+} \to +\infty \quad \text{for a.a. } z \in \widehat{\Omega}.$$

Then by Fatou's lemma, we have

$$\lim_{n\to+\infty} \int_{\widehat{\Omega}} \frac{F(z, u_n)}{\|u_n\|^{p_+}}\mathrm{d}z = +\infty. \tag{3.34}$$

Hypotheses $H_1(i)$, (ii) imply that we can find $c_6 > 0$ such that

$$\frac{F(z, x)}{x^{p_+}} \geqslant -c_6 \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \geqslant 0. \tag{3.35}$$

We have

$$\int_\Omega \frac{F(z, u_n)}{\|u_n\|^{p_+}}\mathrm{d}z = \int_{\widehat{\Omega}} \frac{F(z, u_n)}{\|u_n\|^{p_+}}\mathrm{d}z + \int_{\Omega\setminus\widehat{\Omega}} \frac{F(z, u_n)}{\|u_n\|^{p_+}}\mathrm{d}z \geqslant \int_{\widehat{\Omega}} \frac{F(z, u_n)}{\|u_n\|^{p_+}}\mathrm{d}z - c_7 \quad \forall n \in \mathbb{N}$$

for some $c_7 > 0$ (see (3.35)), so

$$\lim_{n\to+\infty} \int_\Omega \frac{F(z, u_n)}{\|u_n\|^{p_+}}\mathrm{d}z = +\infty \tag{3.36}$$

(see (3.36)). From (3.29), we have

$$-\int_\Omega \frac{1}{\|u_n\|^{p_+-p(z)}}|Dy_n|^{p(z)} - \int_\Omega \frac{1}{\|u_n\|^{p_+-q(z)}}|Dy_n|^{q(z)} + \lambda_n \int_\Omega \frac{f(z, u_n)u_n}{\|u_n\|^{p_+}}\mathrm{d}z = 0 \quad \forall n \in \mathbb{N},$$

so

$$\lambda_n \int_\Omega \frac{f(z, u_n)u_n}{\|u_n\|^{p_+}}\mathrm{d}z \leqslant c_8 \quad \forall n \in \mathbb{N},$$

for some $c_8 > 0$ (see (3.32), recall that $q_+ < p(z) \leqslant p_+$ for all $z \in \overline{\Omega}$), thus

$$\lambda_n \int_\Omega \frac{p_+ F(z, u_n)}{\|u_n\|^{p_+}}\mathrm{d}z - \lambda_n \|\eta\|_1 \leqslant c_8 \quad \forall n \in \mathbb{N}$$

(see hypothesis $H_1(iv)$ and recall that $u_n \geqslant 0$), hence

$$\int_\Omega \frac{p_+ F(z,u_n)}{\|u_n\|^{p_+}}\mathrm{d}z \leqslant \frac{c_8}{\lambda_n} + \|\eta\|_1 \quad \forall n \in \mathbb{N}. \tag{3.37}$$

Comparing (3.36) and (3.37), we have a contradiction.

Next suppose that $y = 0$. We consider the C^1-functional $\varphi_\lambda^* : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda^*(u) = \frac{1}{p_+}\varrho_p(Du) - \lambda \int_\Omega F(z,u)\mathrm{d}z \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

Evidently, we have

$$\varphi_\lambda^* \leqslant \varphi_\lambda \quad \forall \lambda > 0. \tag{3.38}$$

Let $\vartheta_n(t) = \varphi_{\lambda_n}^*(tu_*)$ for all $t \in [0,1]$, all $n \in \mathbb{N}$. We can find $t_n \in [0,1]$ such that

$$\vartheta_n(t_n) = \max_{0\leqslant t\leqslant 1} \vartheta_n(t).$$

Let $\beta \geqslant 1$ and set

$$v_n(z) = (2\beta)^{\frac{1}{p(z)}} y_n(z) \quad \forall n \in \mathbb{N}.$$

Clearly, we have

$$v_n \to 0 \quad \text{in } L^{p(z)}(\Omega)$$

(see (3.33) and recall that $y = 0$), so

$$\int_\Omega F(z,v_n)\mathrm{d}z \to 0 \quad \text{as } n \to +\infty. \tag{3.39}$$

From (3.32), we see that we can find $n_0 \in \mathbb{N}$ such that

$$(2\beta)^{\frac{1}{p(z)}} \frac{1}{\|u_n\|} \leqslant 1 \quad \forall n \geqslant n_0,\, z \in \overline{\Omega}.$$

It follows that

$$\vartheta_n(t_n) \geqslant \vartheta_n\left(\frac{(2\beta)^{\frac{1}{p(z)}}}{\|u_n\|}\right) \quad \forall n \geqslant n_0,\, z \in \overline{\Omega},$$

so

$$\varphi_{\lambda_n}^*(t_n u_n) \geqslant \varphi_{\lambda_n}^*\left((2\beta)^{\frac{1}{p(z)}} y_n\right) = \varphi_{\lambda_n}^*(v_n) \quad \forall n \geqslant n_0,$$

thus

$$\varphi_{\lambda_n}^*(t_n u_n) \geqslant \frac{2\beta}{p_+}\varrho_p(Dy_n) - \int_\Omega F(z,v_n)\mathrm{d}z \quad \forall n \geqslant n_0$$

and hence

$$\varphi_{\lambda_n}^*(t_n u_n) \geqslant \frac{\beta}{p_+} \quad \forall n \geqslant n_1 \geqslant n_0 \tag{3.40}$$

(see (3.39) and Proposition 2.1(a)).

Since $\beta \geqslant 1$ is arbitrary, from (3.40) we infer that

$$\varphi_{\lambda_n}^*(t_n u_n) \to +\infty \quad \text{as } n \to +\infty. \tag{3.41}$$

We have

$$0 \leqslant t_n u_n \leqslant u_n \quad \forall n \in \mathbb{N},$$

so

$$\sigma(z, t_n u_n) \leqslant \sigma(z, u_n) + \eta(z) \quad \text{for a.a. } z \in \Omega, \quad \text{all } n \in \mathbb{N}$$

(see hypothesis $H_1(iii)$), so

$$\int_\Omega \sigma(z, t_n u_n) \mathrm{d}z \leqslant \int_\Omega \sigma(z, u_n) \mathrm{d}z + \|\eta\|_1 \leqslant c_9 \quad \forall n \in \mathbb{N} \tag{3.42}$$

for some $c_9 > 0$ (see (3.31)). We know that

$$\varphi^*_{\lambda_n}(0) = 0 \quad \text{and} \quad \varphi^*_{\lambda_n}(u_n) \leqslant c_5 \quad \forall n \in \mathbb{N} \tag{3.43}$$

(see (3.26) and (3.38)). Then from (3.41) it follows that $t_n \in (0, 1)$ for all $n \geqslant n_2$. Therefore, we can say that

$$0 = t_n \frac{\mathrm{d}}{\mathrm{d}t} \varphi^*_{\lambda_n}(t u_n)|_{t=t_n},$$

so

$$\langle (\varphi^*_{\lambda_n})'(t_n u_n), t_n u_n \rangle = 0$$

(by the chain rule), thus

$$\varrho_p(D(t_n u_n)) + \varrho_q(D(t_n u_n)) - \lambda_n \int_\Omega f(z, t_n u_n)(t_n u_n) \mathrm{d}z = 0 \quad \forall n \geqslant n_2$$

and hence

$$p_+ \varphi^*_{\lambda_n}(t_n u_n) \leqslant c_9 \quad \forall n \geqslant n_2 \tag{3.44}$$

(see (3.42)).

We compare (3.41) and (3.44) and have a contradiction. This proves that the sequence $\{u_n\}_{n \geqslant 1} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded. Recall that

$$A_{p(z)}(u_n) + A_{q(z)}(u_n) = \lambda_n N_f(u_n) \quad \text{in } W_0^{1,p(z)}(\Omega)^* \quad \forall n \in \mathbb{N},$$

with $N_f(u_n)(\cdot) = f(\cdot, u_n(\cdot))$ (the Nemytskii map corresponding to f). From Proposition 2.2, it follows that

$$\lambda_n \|N_f(u_n)\|_* \leqslant c_{10} \quad \forall n \in \mathbb{N}$$

for some $c_{10} > 0$. Since $u_n \geqslant u_1 \in \operatorname{int} C_+$, on account of hypothesis $H_1(iv)$ and since $\lambda_n \to +\infty$, we have

$$\lambda_n \|N_f(u_n)\|_* \to +\infty,$$

a contradiction. This proves that $\lambda^* < +\infty$. □

According to Proposition 3.9, we have

$$(0, \lambda^*) \subseteq \mathcal{L} \subseteq (0, \lambda^*].$$

Proposition 3.10. *If hypotheses H_0, H_1 hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+, \ u_0 \leqslant \hat{u}, \ u_0 \neq \hat{u}.$$

Proof. Let $\lambda, \vartheta \in (0, \lambda^*)$, $\lambda < \vartheta$. We have $\lambda, \vartheta \in \mathcal{L}$. We can find $u_\vartheta \in S_\vartheta \subseteq \operatorname{int} C_+$ and $u_0 \in S_\lambda \subseteq \operatorname{int} C_+$ such that

$$u_\vartheta - u_0 \in \operatorname{int} C_+$$

(see Proposition 3.4). We introduce the Carathéodory function $g(z,x)$ defined by

$$g(z,x)=\begin{cases} f(z,u_0(z)) & \text{if } x\leqslant u_0(z),\\ f(z,x) & \text{if } u_0(z)<x.\end{cases} \tag{3.45}$$

We set

$$G(z,x)=\int_0^x g(z,s)\,\mathrm{d}s$$

and consider the C^1-functional $\psi_\lambda : W^{1,p(z)}(\Omega)\to\mathbb{R}$ defined by

$$\psi_\lambda(u)=\int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,\mathrm{d}z+\int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,\mathrm{d}z-\lambda\int_\Omega G(z,u)\,\mathrm{d}z \quad \forall u\in W^{1,p(z)}(\Omega).$$

Using (3.45) and the anisotropic regularity theory, we obtain

$$K_{\psi_\lambda}\subseteq[u_0)\cap \operatorname{int} C_+. \tag{3.46}$$

We introduce the following truncation of $g(z,\cdot)$

$$\bar{g}(z,x)=\begin{cases} g(z,x) & \text{if } x\leqslant u_0(z),\\ g(z,u_0(z)) & \text{if } u_0(z)<x.\end{cases} \tag{3.47}$$

This is a Carathéodory function. We set

$$\bar{G}(z,x)=\int_0^x \bar{g}(z,s)\,\mathrm{d}s$$

and consider the C^1-functional $\hat{\psi}_\lambda : W^{1,p(z)}(\Omega)\to\mathbb{R}$ defined by

$$\hat{\psi}_\lambda(u)=\int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,\mathrm{d}z+\int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,\mathrm{d}z-\lambda\int_\Omega \bar{G}(z,u)\,\mathrm{d}z \quad \forall u\in W^{1,p(z)}(\Omega).$$

For this functional, we have that

$$K_{\hat{\psi}_\lambda}\subseteq[u_0,u_\vartheta]\cap \operatorname{int} C_+. \tag{3.48}$$

We may assume that

$$K_{\hat{\psi}_\lambda}\cap[u_0,u_\vartheta]=\{u_0\}. \tag{3.49}$$

Otherwise, on account of (3.46) and (3.45), we see that we already have a second positive smooth solution bigger than u_0 and so we are done.

The functional $\hat{\psi}_\lambda$ is coercive (see Proposition 2.1 and (3.47)). Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_0\in W_0^{1,p(z)}(\Omega)$ such that

$$\hat{\psi}_\lambda(\hat{u}_0)=\min_{u\in W^{1,p(z)}(\Omega)}\hat{\psi}_\lambda(u),$$

so $\hat{u}_0\in K_{\hat{\psi}_\lambda}\subseteq[u_0,u_\vartheta]\cap \operatorname{int} C_+$ (see (3.33)).

Note that

$$\psi_\lambda'|_{[u_0,u_\vartheta]}=\hat{\psi}_\lambda'|_{[u_0,u_\vartheta]}$$

(see (3.45) and (3.47)). So, it follows that $\hat{u}_0=u_0$ (see (3.49)).

Since $u_\vartheta-u_0\in \operatorname{int} C_+$, we see that

$$u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \psi_\lambda,$$

so

$$u_0 \text{ is a local } W_0^{1,p(z)}(\overline{\Omega})\text{-minimizer of } \psi_\lambda \tag{3.50}$$

(see Gasiński-Papageorgiou [6] and Tan-Fang [30]).

From (3.46), we see that we may assume that K_{ψ_λ} is finite (otherwise we already have infinity of positive smooth solutions bigger than u_0 and so we are done). Then on account of (3.50) and using Theorem 5.7.6 of Papageorgiou-Rădulescu-Repovš [33, p. 449], we can find $\varrho \in (0, 1)$ small such that

$$\psi_\lambda(u_0) < \inf\{\psi_\lambda(u) : \|u - u_0\| = \varrho\} = m_\lambda. \tag{3.51}$$

Also, if $u \in \operatorname{int} C_+$, then from (3.45) and hypothesis $H_1(ii)$ we have that

$$\psi_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.52}$$

Claim. ψ_λ satisfies the Cerami condition.

Consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p(z)}(\Omega)$ such that

$$|\psi_\lambda(u_n)| \leqslant c_{11} \quad \forall n \in \mathbb{N}, \tag{3.53}$$

for some $c_{11} > 0$, so

$$(1 + \|u_n\|)\psi_\lambda'(u_n) \to 0 \quad \text{in } W_0^{1,p(z)}(\Omega)^* \quad \text{as } n \to +\infty. \tag{3.54}$$

From (3.54), we have

$$\left| \langle A_{p(z)}(u_n), h\rangle + \langle A_{q(z)}(u_n), h\rangle - \lambda \int_\Omega g(z, u_n) h dz \right| \leqslant \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \forall h \in W_0^{1,p(z)}(\Omega), \tag{3.55}$$

with $\varepsilon_n \to 0^+$. In (3.55), we use $h = -u_n^- \in W_0^{1,p(z)}(\Omega)$ and obtain

$$\varrho_p(Du_n^-) + \varrho_q(Du_n^-) \leqslant c_{12} \quad \forall n \in \mathbb{N},$$

for some $c_{12} > 0$ (see (3.45)), so

$$\text{the sequence } \{u_n^-\}_{n\geqslant 1} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded} \tag{3.56}$$

(see Proposition 2.1).

Next in (3.55) we choose $h = u_n^+ \in W^{1,p(z)}(\Omega)$. Then

$$-\varrho_p(Du_n^+) - \varrho_q(Du_n^+) + \lambda \int_\Omega g(z, u_n^+) u_n^+ dz \leqslant \varepsilon_n \quad \forall n \in \mathbb{N},$$

so

$$-\varrho_p(Du_n^+) - \varrho_q(Du_n^+) + \lambda \int_\Omega f(z, u_n^+) u_n^+ dz \leqslant c_{13} \quad \forall n \in \mathbb{N}, \tag{3.57}$$

for some $c_{13} > 0$.

From (3.53), (3.56) and (3.45), we have

$$\varrho_p(Du_n^+) + \varrho_q(Du_n^+) - \lambda \int_\Omega p_+ F(z, u_n^+) dz \leqslant c_{14} \quad \forall n \in \mathbb{N}, \tag{3.58}$$

for some $c_{14} > 0$.

We add (3.57) and (3.58) and obtain

$$\lambda \int_\Omega \sigma(z, u_n^+) dz \leqslant c_{15} \quad \forall n \in \mathbb{N}, \tag{3.59}$$

for some $c_{15} > 0$.

Using (3.59) and reasoning as in the proof of Proposition 3.9 (see the part of the proof after (3.31) up to (3.44)), we obtain that

$$\text{the sequence } \{u_n^+\} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded.} \tag{3.60}$$

Then (3.50) and (3.60) imply that

$$\text{the sequence } \{u_n\} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p(z)}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^r(\Omega) \quad \text{as } n \to +\infty. \tag{3.61}$$

In (3.55), we test with $h = u_n - u \in W_0^{1,p(z)}(\Omega)$ and pass to the limit as $n \to +\infty$. As in the proof of Proposition 3.7, we obtain

$$u_n \to u \quad \text{in } W_0^{1,p(z)}(\Omega) \quad \text{as } n \to +\infty$$

(see (3.24)), so ψ_λ satisfies the Cerami condition. This proves the Claim.

Then (3.51), (3.52) and the Claim permit the use of the mountain pass theorem and find $\hat{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\hat{u} \in K_{\psi_\lambda} \subseteq [u_0) \cap \operatorname{int} C_+ \quad \text{and} \quad m_\lambda \leqslant \psi_\lambda(\hat{u}) \tag{3.62}$$

(see (3.46) and (3.51)).

From (3.62), (3.51) and (3.45), we conclude that $\hat{u} \in \operatorname{int} C_+$ is a positive solution of (P_λ), $u_0 \leqslant \hat{u}$, $u_0 \neq \hat{u}$. □

It remains to decide what happens with critical parameter value $\lambda^* < +\infty$.

Proposition 3.11. *If hypotheses H_0, H_1 hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Let $\lambda_n \in (0, \lambda^*)$, $n \in \mathbb{N}$ be such that $\lambda_n \nearrow \lambda^*$. We can find $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$ nondecreasing such that

$$\varphi_{\lambda_n}(u_n) \leqslant c_{16} \quad \forall n \in \mathbb{N}, \tag{3.63}$$

for some $c_{16} > 0$, so

$$\varphi'_{\lambda_n}(u_n) = 0 \quad \forall n \in \mathbb{N}. \tag{3.64}$$

Using (3.63), (3.64) as in the proof of Proposition 3.9, first we obtain that the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded and then via Proposition 2.2, at least for a subsequence, we have

$$u_n \to u^* \quad \text{in } W_0^{1,p(z)}(\Omega). \tag{3.65}$$

From (3.64) and (3.65), in the limit as $n \to +\infty$, we obtain

$$\langle A_{p(z)}(u^*), h\rangle + \langle A_{q(z)}(u^*), h\rangle = \lambda^* \int_\Omega f(z, u^*) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so $u_1 \leqslant u^*$. Therefore, $u^* \in S_{\lambda^*} \subseteq \operatorname{int} C_+$ and so $\lambda^* \in \mathcal{L}$. □

We conclude that

$$\mathcal{L} = (0, \lambda^*].$$

So, summarizing our findings for problem (P_λ), we can state the following bifurcation-type theorem.

Theorem 3.12. *If hypotheses* H_0, H_1 hold, then there exists $\lambda^* > 0$ such that

(*a*) *for all* $\lambda \in (0, \lambda^*)$ *problem* (P_λ) *has at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+, u_0 \leqslant \hat{u}, u_0 \neq \hat{u};$$

(*b*) *for* $\lambda = \lambda^*$ *problem* (P_λ) *has at least one positive solution*

$$u^* \in \operatorname{int} C_+;$$

(*c*) *for all* $\lambda > \lambda^*$ *problem* (P_λ) *has no positive solutions;*

(*d*) *for all* $\lambda \in (0, \lambda^*]$ *problem* (P_λ) *has a smallest (minimal) positive solution* $u_\lambda^* \in \operatorname{int} C_+$ *and the map* $\lambda \mapsto u_\lambda^*$ *from* $\mathcal{L} = (0, \lambda^*]$ *into* $C_0^1(\overline{\Omega})$ *is strictly increasing and left continuous.*

Acknowledgements: The authors wish to thank the two knowledgeable referees for their corrections and remarks.

References

[1] L. Gasiński and N. S. Papageorgiou, *Bifurcation-type results for nonlinear parametric elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), no. 3, 595–623.

[2] L. Gasiński and N. S. Papageorgiou, *Positive solutions for the Robin p-Laplacian problem with competing nonlinearities*, Adv. Calc. Var. **12** (2019), no. 1, 31–56.

[3] D. Mugnai and N. S. Papageorgiou, *Wang's multiplicity result for superlinear (p,q)-equations without the Ambrosetti-Rabinowitz condition*, Trans. Amer. Math. Soc. **366** (2014), no. 9, 4919–4937.

[4] N. S. Papageorgiou and V. D. Rădulescu, *Coercive and noncoercive nonlinear Neumann problems with indefinite potential*, Forum Math. **28** (2016), no. 3, 545–571.

[5] N. S. Papageorgiou and A. Scapellato, *Constant sign and nodal solutions for parametric (p,2)-equations*, Adv. Nonlinear Anal. **9** (2020), no. 1, 449–478.

[6] L. Gasiński and N. S. Papageorgiou, *Anisotropic nonlinear Neumann problems*, Calc. Var. Partial Differential Equations **42** (2011), no. 3–4, 323–354.

[7] N. S. Papageorgiou, V. D. Rădulescu, and D. D. Repovš, *Anisotropic equations with indefinite potential and competing nonlinearities*, Nonlinear Anal. **201** (2020), 111861, DOI: 10.1016/j.na.2020.111861.

[8] N. S. Papageorgiou and C. Vetro, *Superlinear (p(z),q(z))-equations*, Complex Var. Elliptic Equ. **64** (2019), no. 1, 8–25.

[9] A. Bahrouni, V. Rădulescu, and D. Repovš, *A weighted anisotropic variant of the Caffarelli-Kohn-Nirenberg inequality and applications*, Nonlinearity **31** (2018), no. 4, 1516–1534.

[10] N. S. Papageorgiou, C. Vetro, and F. Vetro, *Continuous spectrum for a two phase eigenvalue problem with an indefinite and unbounded potential*, J. Differential Equations **268** (2020), no. 8, 4102–4118.

[11] C. Vetro, *Parametric and nonparametric A-Laplace problems: Existence of solutions and asymptotic analysis*, Asymptotic Analysis (2020), DOI: 10.3233/ASY-201612.

[12] C. Vetro, *Weak solutions to Dirichlet boundary value problem driven by p(x)-Laplacian-like operator*, Electron. J. Qual. Theory Differ. Equ. **2017** (2017), no. 98, 1–10, DOI: 10.14232/ejqtde.2017.1.98.

[13] C. Vetro and F. Vetro, *On problems driven by the (p(·), q(·))-Laplace operator*, Mediterr. J. Math. **17** (2020), 24, DOI: 10.1007/s00009-019-1448-1.

[14] F. Vetro, *Infinitely many solutions for mixed Dirichlet-Neumann problems driven by the (p,q)-Laplace operator*, Filomat **33** (2019), no. 14, 4603–4611.

[15] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics, 1748*, Springer-Verlag, Berlin, 2000.

[16] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents,* Lecture Notes in Mathematics, vol. 2017, Springer, Heidelberg, 2011.

[17] P. Harjulehto, P. Hästö, Ú. V. Lê, and M. Nuortio, *Overview of differential equations with non-standard growth*, Nonlinear Anal. **72** (2010), no. 12, 4551–4574.

[18] V. Rădulescu and D. Repovš, *Partial Differential Equations with Variable Exponents*, CRC Press, Boca Raton, FL, 2015.

[19] A. Bahrouni, V. Rădulescu, and D. Repovš, *Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves*, Nonlinearity **32** (2019), no. 7, 2481–2495.

[20] V. Benci, P. D'Avenia, D. Fortunato, and L. Pisani, *Solitons in several space dimensions: Derrick's problem and infinitely many solutions*, Arch. Ration. Mech. Anal. **154** (2000), no. 4, 297–324.

[21] L. Cherfils and Y. Il'yasov, *On the stationary solutions of generalized reaction diffusion equations with p&q-Laplacian*, Commun. Pure Appl. Anal. **4** (2005), no. 1, 9–22.

[22] V. V. Zhikov, *On variational problems and nonlinear elliptic equations with nonstandard growth conditions*, J. Math. Sci. (N.Y.) **173** (2011), no. 5, 463–570.

[23] S. A. Marano and S. J. N. Mosconi, *Some recent results on the Dirichlet problem for* (p,q)*-Laplace equations*, Discrete Contin. Dyn. Syst. Ser. S **11** (2018), no. 2, 279–291.

[24] V. Rădulescu, *Isotropic and anisotropic double-phase problems: old and new*, Opuscula Math. **39** (2019), no. 2, 259–279.

[25] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381.

[26] G. Li and C. Yang, *The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti-Rabinowitz condition*, Nonlinear Anal. **72** (2010), no. 12, 4602–4613.

[27] L. Gasiński and N. S. Papageorgiou, *Nonlinear Analysis*, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[28] X. Fan and D. Zhao, *A class of De Giorgi type and Hölder continuity*, Nonlinear Anal. **36** (1999), no. 3, 295–318.

[29] N. Fukagai and K. Narukawa, *On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems*, Ann. Mat. Pura Appl. **186** (2007), no. 3, 539–564.

[30] Z. Tan and F. Fang, *Orlicz-Sobolev versus Hölder local minimizer and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl. **402** (2013), no. 1, 348–370.

[31] G. M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), no. 2–3, 311–361.

[32] Q. Zhang, *A strong maximum principle for differential equations with nonstandard* $p(x)$*-growth conditions*, J. Math. Anal. Appl. **312** (2005), no. 1, 24–32.

[33] N. S. Papageorgiou, V. D. Rădulescu, and D. D. Repovš, *Nonlinear Analysis – Theory and Methods*, Vol. 1, Springer, Switzerland, 2019.

[34] P. Takáč and J. Giacomoni, *A* $p(x)$*-Laplacian extension of the Díaz-Saa inequality and some applications*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 1, 205–232.

[35] N. S. Papageorgiou, V. D. Rădulescu, and D. D. Repovš, *Positive solutions for perturbations of the Robin eigenvalue problem plus an indefinite potential*, Discrete Contin. Dyn. Syst. **37** (2017), no. 5, 2589–2618.

[36] M. E. Filippakis and N. S. Papageorgiou, *Multiple constant sign and nodal solutions for nonlinear elliptic equations with the p-Laplacian*, J. Differential Equations **245** (2008), no. 7, 1883–1922.

[37] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis. Vol. I, Theory*, Kluwer Academic Publishers, Dordrecht, 1997.