Xin and Hu *Boundary Value Problems* (2020) 2020:143
https://doi.org/10.1186/s13661-020-01441-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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Weak and strong singularities problems to Liénard equation

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Abstract

This paper is devoted to an investigation of the existence of a positive periodic solution for the following singular Liénard equation:

$$x'' + f(x(t))x'(t) + a(t)x = \frac{b(t)}{x^\alpha} + e(t),$$

where the external force $e(t)$ may change sign, α is a constant and $\alpha > 0$. The novelty of the present article is that for the first time we show that weak and strong singularities enables the achievement of a new existence criterion of positive periodic solution through an application of the Manásevich–Mawhin continuation theorem. Recent results in the literature are generalized and significantly improved, and we give the existence interval of periodic solution of this equation. At last, two examples and numerical solution (phase portraits and time portraits of periodic solutions of the example) are given to show applications of the theorem.

MSC: 34B16; 34B18; 34C25

Keywords: Weak and strong singularities; Liénard equation; Positive periodic solution

1 Introduction

The main purpose of this paper is to consider the existence of a periodic solution for the Liénard equation with weak and strong singularities of repulsive type,

$$x'' + f(x(t))x'(t) + a(t)x = \frac{b(t)}{x^\alpha} + e(t), \tag{1.1}$$

where $a, b \in C(\mathbb{R}, (0, +\infty))$ are ω-periodic functions, $f \in C(\mathbb{R}, \mathbb{R})$, the external force $e \in C(\mathbb{R}, \mathbb{R})$ is an ω-periodic function. Moreover, note that when $f(x(t)) \equiv 0$, Eq. (1.1) becomes

$$x'' + a(t)x = \frac{b(t)}{x^\alpha} + e(t). \tag{1.2}$$

In 1987, Lazer and Solimini [1] investigated the following second-order differential equation with singularity of repulsive type:

$$x'' = \frac{1}{x^\alpha} + h(t),$$

and obtained the result that if the external force $h(t)$ was continuous and ω-periodic, then for all $\alpha > 0$ a positive periodic solution existed if and only if the external force $h(t)$ has a positive mean value. We say the equation to obey the strong force condition if $\alpha \geq 1$ and the weak force condition if $0 < \alpha < 1$.

Lazer and Solimini's work has attracted the attention of many scholars in singular equations. More recently, the Poincaré–Birkhoff twist theorem [2–4], Schauder's fixed point theorem [5–8], the Leray–Schauder alternative principle [9–11], coincidence degree theory [12–15],the Krasnoselskii fixed point theorem in cones [16, 17] and Leray–Schauder degree theory [18, 19] have been employed to discuss the existence of a positive periodic solution of singular equations.

Among these papers, there have been published some results on Eq. (1.2) (see [5, 6, 8, 10, 17]). Chu et al. [10] in 2007 discussed the existence of a positive periodic solution for Eq. (1.2) if the external force $e(t) \geq 0$ and $\|a\| := \max_{t\in[0,\omega]} |a(t)| < \frac{\pi^2}{\omega^2}$. Their results were based on a nonlinear alternative principle of Leray–Schauder and are applicable to the case of a strong singularity and the case of a weak singularity. Afterwards, Torres [8] proved Eq. (1.2) in the cases of weak and strong singularities had at least one positive periodic solution if the external force $e(t) > 0$ and $\|a\| < \frac{\pi^2}{\omega^2}$. Moreover, the author obtained the result that there was one positive periodic solution for Eq. (1.2) in the case of a weak singularity if one of the following conditions holds:

(i) $e(t) \equiv 0$ and $\|a\| < \frac{\pi^2}{\omega^2}$; or (ii) $e(t) < 0$ and $\|a\| < \frac{\pi^2}{\omega^2}$.

Wang [17] in 2010 improved the above result and presented a new assumption, which is weaker than the singular condition in [8]. The author obtained the result that Eq. (1.2) in the cases that weak and strong singularities have at least one positive periodic solution if and only if one of the following conditions holds:

(i) $e(t) \geq 0$ and $\|a\| < \frac{\pi^2}{\omega^2}$; or (ii) $e(t) < 0$ and $\|a\| < \frac{\pi^2}{\omega^2}$.

The proof of their results was based on the Krasnoselskii fixed point theorem in a cone.

All the aforementioned results are related to Eq. (1.2) with the external force $e(t)$ not changing sign. Naturally, a new question arises: how may Eq. (1.1) with weak and strong singularities work on the external force $e(t)$ changing sign? In this paper, we fill the gap and provide sufficient conditions for the existence of a positive periodic solution for Eq. (1.1) with weak and strong singularities, where the external force $e(t)$ may change sign, α is a constant and $\alpha > 0$. By applications of the Manásevich–Mawhin continuation theorem [20, Theorem 3.1], we obtain the following conclusion.

Theorem 1.1 *Assume that the following conditions are satisfied*:

(H_1) $\int_0^\omega e(t)\,dt = 0$;

(H_2) $\|a\| < \frac{\pi^2}{\omega^2}$;

(H_3)

$$\sigma_* > \left(\frac{\omega^{\frac{1}{2}}}{2}\left(\frac{\pi^2\omega(\|a\|(\sigma^*)^{\frac{2}{1+\alpha}} + \|e\|(\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2 - \|a\|\omega^2}\right)^{\frac{1}{2}} + \frac{\pi\omega^2(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}} + \|e\|)}{2(\pi^2 - \|a\|\omega^2)}\right)^{1+\alpha},$$

where $\sigma^* := \frac{b^*}{a_*}$, $\sigma_* := \frac{b_*}{a^*}$, $b^* := \max_{t\in[0,\omega]} b(t)$, $b_* := \min_{t\in[0,\omega]} b(t)$.

Then Eq. (1.1) *has at least one positive periodic solution x with*

$$x \in \left((\sigma_*)^{\frac{1}{1+\alpha}} - \frac{\omega^{\frac{1}{2}}}{2} \left(\frac{\pi^2 \omega (\|a\| (\sigma^*)^{\frac{2}{1+\alpha}} + \|e\| (\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2 - \|a\| \omega^2} \right)^{\frac{1}{2}} - \frac{\pi \omega^2 (2\|a\| (\sigma^*)^{\frac{1}{1+\alpha}} + \|e\|)}{2(\pi^2 - \|a\| \omega^2)}, \right.$$
$$\left. (\sigma^*)^{\frac{1}{1+\alpha}} + \frac{\omega^{\frac{1}{2}}}{2} \left(\frac{\pi^2 \omega (\|a\| (\sigma^*)^{\frac{2}{1+\alpha}} + \|e\| (\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2 - \|a\| \omega^2} \right)^{\frac{1}{2}} + \frac{\pi \omega^2 (2\|a\| (\sigma^*)^{\frac{1}{1+\alpha}} + \|e\|)}{2(\pi^2 - \|a\| \omega^2)} \right).$$

Remark 1.1 The techniques used are quite different from that in [5, 8, 10, 17] and our results are more general than those in [5, 8, 10, 17] in two aspects. We first obtain the existence of a positive periodic solution for equation (1.1) with weak and strong singularities if the external force $e(t)$ may change sign. Secondly, we give the existence interval of positive periodic solution of Eq. (1.1).

In the following, we consider the existence of a periodic solution for Eq. (1.1) without the external force $e(t)$.

Corollary 1.1 *Assume that* (H_2) *holds. Furthermore, suppose the following conditions are satisfied*:

(H_1') $e(t) = 0$;

(H_3') $\sigma_* > (\frac{\omega^{\frac{1}{2}}}{2}(\frac{\pi^2\omega\|a\|(\sigma^*)^{\frac{2}{1+\alpha}}}{\pi^2-\|a\|\omega^2})^{\frac{1}{2}} + \frac{\pi\omega^2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}}}{(\pi^2-\|a\|\omega^2)})^{1+\alpha}$.

Then Eq. (1.1) *has at least one positive periodic solution x with*

$$x \in \left((\sigma_*)^{\frac{1}{1+\alpha}} - \frac{\omega^{\frac{1}{2}}}{2} \left(\frac{\pi^2 \omega \|a\| (\sigma^*)^{\frac{2}{1+\alpha}}}{\pi^2 - \|a\| \omega^2} \right)^{\frac{1}{2}} - \frac{\pi \omega^2 \|a\| (\sigma^*)^{\frac{1}{1+\alpha}}}{(\pi^2 - \|a\| \omega^2)}, \right.$$
$$\left. (\sigma^*)^{\frac{1}{1+\alpha}} + \frac{\omega^{\frac{1}{2}}}{2} \left(\frac{\pi^2 \omega \|a\| (\sigma^*)^{\frac{2}{1+\alpha}}}{\pi^2 - \|a\| \omega^2} \right)^{\frac{1}{2}} + \frac{\pi \omega^2 \|a\| (\sigma^*)^{\frac{1}{1+\alpha}}}{(\pi^2 - \|a\| \omega^2)} \right).$$

Obviously, the condition (H_3) (or (H_3')) is hard restrictive for the existence of a positive periodic solution to Eq. (1.1). In the following, we study the existence of a positive periodic solution for Eq. (1.1) with strong singularity (i.e. $\alpha \geq 1$) if conditions (H_1) and (H_2) are satisfied.

Theorem 1.2 *Assume that conditions* (H_1) *and* (H_2) *hold. Furthermore, suppose the following condition is satisfied*:

(H_4) $b(t) \equiv b$ *and b is a positive constant.*

Then Eq. (1.1) *has at least one positive periodic solution if* $\alpha \geq 1$.

2 Proof of theorems

We first recall the Sobolev inequality recently proved in [21, P. 357] and the topological degree theorem by Mawhin [20].

Lemma 2.1 (Sobolev inequality; see [21]) *Let* $u \in W^{1,p}(\mathbb{R})$ *and* $u(0) = u(\omega) = 0$. *Then we have*

$$S(\sigma)\omega^{-1-2/\sigma} \|x\|_\sigma^2 \leq \|x'\|_2^2, \tag{2.1}$$

where $S(\sigma)$ is the Sobolev constant, σ is a positive constant and $\sigma > 1$, $S(\sigma) = \frac{4(1+\frac{\sigma}{2})^2 B^2(\frac{1}{\sigma},\frac{1}{2})}{\sigma^2(1+\frac{1}{\sigma})}$ and $B(\cdot,\cdot)$ is the Beta function.

Remark 2.1 When $\sigma = 2$, we get $S(2) = \pi^2$.

Next, we investigate a family of (1.1) as follows:

$$x''(t) + \lambda\left(f\bigl(x(t)\bigr)x'(t) + a(t)x(t) - \frac{b(t)}{x^\alpha(t)}\right) = \lambda e(t), \quad \lambda \in (0,1]. \tag{2.2}$$

Using [20, Theorem 3.1], we obtain the following conclusion.

Lemma 2.2 *Assume that there exist positive constants E_1, E_2, E_3 and $E_1 < E_2$ such that the following conditions hold*:

(1) *Each possible periodic solution x to Eq. (2.2) such that $E_1 < x(t) < E_2$, $\forall t \in [0,\omega]$ and $\|x'\| < E_3$.*

(2) *Each possible solution C to the equation*

$$\int_0^\omega \left(a(t)C - \frac{b(t)}{C^\alpha}\right) dt = 0$$

satisfies $C \in (E_1, E_2)$.

(3) *We have*

$$\int_0^\omega \left(a(t)E_1 - \frac{b(t)}{E_1^\alpha}\right) dt \cdot \int_0^\omega \left(a(t)E_2 - \frac{b(t)}{E_2^\alpha}\right) dt < 0.$$

Then Eq. (1.1) has at least one positive periodic solution.

We investigate the existence of a periodic solution for Eq. (1.1) with weak and strong singularities.

Proof of Theorem 1.1 Integrating Eq. (2.2) from 0 to ω, we get

$$\int_0^\omega \left(a(t)x(t) - \frac{b(t)}{x^\alpha(t)}\right) dt = 0. \tag{2.3}$$

In view of the mean value theorem of integrals, we know that there exists a point $\xi \in (0,\omega)$ such that

$$a(\xi)x(\xi) = \frac{b(\xi)}{x^\alpha(\xi)},$$

since $\int_0^\omega x''(t)\,dt = 0$ and $\int_0^\omega e(t)\,dt = 0$ from condition (H_1). Furthermore, we deduce

$$(\sigma_*)^{\frac{1}{1+\alpha}} \le x(\xi) \le \bigl(\sigma^*\bigr)^{\frac{1}{1+\alpha}}. \tag{2.4}$$

Multiplying both sides of Eq. (2.2) by $x(t)$ and integrating on the interval $[0,\omega]$, we obtain

$$\begin{aligned}&\int_0^\omega x''(t)x(t)\,dt+\lambda\int_0^\omega f\bigl(x(t)\bigr)x(t)x'(t)\,dt+\lambda\int_0^\omega a(t)\bigl|x(t)\bigr|^2\,dt\\&\quad=\lambda\int_0^\omega b(t)x^{1-\alpha}(t)\,dt+\lambda\int_0^\omega e(t)x(t)\,dt.\end{aligned}\tag{2.5}$$

Substituting $\int_0^\omega x''(t)x'(t)\,dt=-\int_0^\omega |x'(t)|^2\,dt$ and $\int_0^T f(x(t))x(t)x'(t)\,dt=0$ into Eq. (2.5), applying the Hölder inequality, we have

$$\begin{aligned}\int_0^\omega\bigl|x'(t)\bigr|^2\,dt&=\lambda\int_0^\omega a(t)\bigl|x(t)\bigr|^2\,dt-\lambda\int_0^\omega b(t)x^{1-\alpha}(t)\,dt-\lambda\int_0^\omega e(t)x(t)\,dt\\&\le\int_0^\omega\bigl|a(t)\bigr|\bigl|x(t)\bigr|^2\,dt+\int_0^\omega\bigl|e(t)\bigr|\bigl|x(t)\bigr|\,dt\\&\le\|a\|\int_0^\omega\bigl|x(t)\bigr|^2\,dt+\|e\|\omega^{\frac12}\biggl(\int_0^T\bigl|x'(t)\bigr|^2\,dt\biggr)^{\frac12},\end{aligned}\tag{2.6}$$

since $b(t)>0$ and $x(t)>0$. Define $u(t):=x(t+\xi)-x(\xi)$, where ξ is as in Eq. (2.4), then $u(0)=u(\omega)=0$. Using Eq. (2.4), Lemma 2.1 and the Minkowski inequality, we deduce

$$\begin{aligned}\biggl(\int_0^\omega\bigl|x(t)\bigr|^2\,dt\biggr)^{\frac12}&=\biggl(\int_0^\omega\bigl|u(t)+x(\xi)\bigr|^2\,dt\biggr)^{\frac12}\\&\le\biggl(\int_0^\omega\bigl|u(t)\bigr|\,dt\biggr)^{\frac12}+\biggl(\int_0^\omega\bigl|x(\xi)\bigr|\,dt\biggr)^{\frac12}\\&\le\frac{\omega}{\sqrt{S(2)}}\biggl(\int_0^\omega\bigl|u'(t)\bigr|^2\,dt\biggr)^{\frac12}+\bigl(\sigma^*\bigr)^{\frac{1}{1+\alpha}}\omega^{\frac12}\\&\le\frac{\omega}{\pi}\biggl(\int_0^\omega\bigl|x'(t)\bigr|^2\,dt\biggr)^{\frac12}+\bigl(\sigma^*\bigr)^{\frac{1}{1+\alpha}}\omega^{\frac12},\end{aligned}\tag{2.7}$$

since $u'(t)=x'(t)$ and $S(2)=\pi^2$ from Remark 2.1. Substituting Eqs. (2.7) into (2.6), we arrive at

$$\begin{aligned}\int_0^\omega\bigl|x'(t)\bigr|^2\,dt&\le\|a\|\biggl(\frac{\omega}{\pi}\biggl(\int_0^\omega\bigl|x'(t)\bigr|^2\,dt\biggr)^{\frac12}+\bigl(\sigma^*\bigr)^{\frac{1}{1+\alpha}}\omega^{\frac12}\biggr)^2\\&\quad+\|e\|\omega^{\frac12}\frac{\omega}{\pi}\biggl(\biggl(\int_0^\omega\bigl|x'(t)\bigr|^2\,dt\biggr)^{\frac12}+\bigl(\sigma^*\bigr)^{\frac{1}{1+\alpha}}\omega^{\frac12}\biggr)\\&=\|a\|\frac{\omega^2}{\pi^2}\int_0^\omega\bigl|x'(t)\bigr|^2\,dt+\frac{\omega^{\frac32}}{\pi}\bigl(2\|a\|\bigl(\sigma^*\bigr)^{\frac{1}{1+\alpha}}+\|e\|\bigr)\biggl(\int_0^\omega\bigl|x'(t)\bigr|^2\,dt\biggr)^{\frac12}\\&\quad+\bigl(\|a\|\bigl(\sigma^*\bigr)^{\frac{2}{1+\alpha}}+\|e\|\bigl(\sigma^*\bigr)^{\frac{1}{1+\alpha}}\bigr)\omega\\&=\|a\|\frac{\omega^2}{\pi^2}\int_0^\omega\bigl|x'(t)\bigr|^2\,dt+N_1\biggl(\int_0^\omega\bigl|x'(t)\bigr|^2\,dt\biggr)^{\frac12}+N_2,\end{aligned}$$

where $N_1 := \frac{\omega^{\frac{3}{2}}}{\pi}(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}} + \|e\|)$ and $N_2 := (\|a\|(\sigma^*)^{\frac{2}{1+\alpha}} + \|e\|(\sigma^*)^{\frac{1}{1+\alpha}})\omega$. From condition (H_2), we see that

$$\left(1 - \|a\|\frac{\omega^2}{\pi^2}\right)\int_0^\omega \left|x'(t)\right|^2 dt \le N_1\left(\int_0^\omega \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} + N_2.$$

It is clear that

$$\int_0^\omega \left|x'(t)\right|^2 dt \le \frac{\pi^2 N_1}{\pi^2 - \|a\|\omega^2}\left(\int_0^\omega \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} + \frac{\pi^2 N_2}{\pi^2 - \|a\|\omega^2}.$$

Furthermore, we obtain

$$\left(\left(\int^\omega \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} - \frac{\pi^2 N_1}{2\pi^2 - 2\|a\|\omega^2}\right)^2 \le \frac{\pi^2 N_2}{\pi^2 - \|a\|\omega^2} + \frac{\pi^4 N_1^2}{(2\pi^2 - 2\|a\|\omega^2)^2}.$$

Therefore, the above inequality implies

$$\begin{aligned}
&\left(\int_0^\omega \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\quad\le \left(\frac{\pi^2 N_2}{\pi^2 - \|a\|\omega^2} + \frac{\pi^4 N_1^2}{(2\pi^2 - 2\|a\|\omega^2)^2}\right)^{\frac{1}{2}} + \frac{\pi^2 N_1}{2\pi^2 - 2\|a\|\omega^2} \\
&\quad= \left(\frac{\pi^2\omega(\|a\|(\sigma^*)^{\frac{2}{1+\alpha}} + \|e\|(\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2 - \|a\|\omega^2} + \frac{\pi^2\omega^3(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}} + \|e\|)^2}{(2\pi^2 - 2\|a\|\omega^2)^2}\right)^{\frac{1}{2}} \\
&\qquad+ \frac{\pi\omega^{\frac{3}{2}}(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}} + \|e\|)}{2\pi^2 - 2\|a\|\omega^2} \\
&\quad\le \left(\frac{\pi^2\omega(\|a\|(\sigma^*)^{\frac{2}{1+\alpha}} + \|e\|(\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2 - \|a\|\omega^2}\right)^{\frac{1}{2}} + \frac{\pi\omega^{\frac{3}{2}}(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}} + \|e\|)}{\pi^2 - \|a\|\omega^2} := M_1' \qquad (2.8)
\end{aligned}$$

where the lase inequality holds because of a classical inequality, i.e.,

$$(a+b)^k \le a^k + b^k, \quad \text{for } k \in (0,1), a, b \in (0,+\infty).$$

From Eqs. (2.4), (2.8) and the Hölder inequality, we deduce

$$\begin{aligned}
x(t) &= \frac{1}{2}\bigl(x(t) + x(t-\omega)\bigr) \\
&= \frac{1}{2}\left(x(\xi) + \int_\xi^t x'(s)\,ds + x(\xi) - \int_{t-\omega}^\xi x'(s)\,ds\right) \\
&\le x(\xi) + \frac{1}{2}\left(\int_\xi^t \left|x'(s)\right| ds + \int_{t-\omega}^\xi \left|x'(s)\right| ds\right) \\
&= x(\xi) + \frac{1}{2}\int_{t-\omega}^t \left|x'(s)\right| ds \\
&\le (\sigma^*)^{\frac{1}{1+\alpha}} + \frac{1}{2}\int_0^\omega \left|x'(t)\right| dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sigma^*\right)^{\frac{1}{1+\alpha}} + \frac{\omega^{\frac{1}{2}}}{2}\left(\int_0^\omega \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\leq \left(\sigma^*\right)^{\frac{1}{1+\alpha}} + \frac{\omega^{\frac{1}{2}}}{2}\left(\frac{\pi^2\omega(\|a\|(\sigma^*)^{\frac{2}{1+\alpha}} + \|e\|(\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2 - \|a\|\omega^2}\right)^{\frac{1}{2}} \\
&\quad + \frac{\pi\omega^2(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}} + \|e\|)}{2(\pi^2 - \|a\|\omega^2)} := M_1.
\end{aligned}
\tag{2.9}
$$

On the other hand, from Eqs. (2.4), (2.8) and (2.9), we get

$$
\begin{aligned}
x(t) &= x(\xi) + \frac{1}{2}\int_{t-\omega}^t x'(s)\,ds \\
&\geq (\sigma_*)^{\frac{1}{1+\alpha}} - \frac{1}{2}\left(\int_\xi^t \left|x'(s)\right| ds + \int_{t-\omega}^\xi \left|x'(s)\right| ds\right) \\
&\geq (\sigma_*)^{\frac{1}{1+\alpha}} - \frac{1}{2}\int_0^\omega \left|x'(t)\right| ds \\
&\geq (\sigma_*)^{\frac{1}{1+\alpha}} - \frac{\omega^{\frac{1}{2}}}{2}\left(\int_0^\omega \left|x'(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\geq (\sigma_*)^{\frac{1}{1+\alpha}} - \frac{\omega^{\frac{1}{2}}}{2}\left(\frac{\pi^2\omega(\|a\|(\sigma^*)^{\frac{2}{1+\alpha}} + \|e\|(\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2 - \|a\|\omega^2}\right)^{\frac{1}{2}} \\
&\quad - \frac{\pi\omega^2(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}} + \|e\|)}{2(\pi^2 - \|a\|\omega^2)} := M_2 > 0,
\end{aligned}
\tag{2.10}
$$

since $\sigma_* > (\frac{\omega^{\frac{1}{2}}}{2}(\frac{\pi^2\omega(\|a\|(\sigma^*)^{\frac{2}{1+\alpha}}+\|e\|(\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2-\|a\|\omega^2})^{\frac{1}{2}} + \frac{\pi\omega^2(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}}+\|e\|)}{2(\pi^2-\|a\|\omega^2)})^{1+\alpha}$ from condition (H_3).

Next, we are going to obtain a uniform bound on $x'(t)$. In fact, in view of $x(0) = x(\omega)$, there exists a point $t_1 \in (0,\omega)$ such that $x'(t_1) = 0$, from Eqs. (2.2), (2.9) and (2.10), it is easy to see that

$$
\begin{aligned}
\left\|x'\right\| &= \max_{t\in[0,\omega]}\left\{\left|x'(t)\right|\right\} \\
&= \max_{t\in[t_1,t_1+\omega]}\left\{\left|\int_{t_1}^t \left(x''(s)\right) ds\right|\right\} \\
&\leq \int_0^\omega \left|f\left(x(t)\right)\right|\left|x'(t)\right| dt + \int_0^\omega \left|a(t)\right|\left|x(t)\right| dt + \int_0^\omega \left|\frac{b(t)}{x^\alpha(t)}\right| dt + \int_0^\omega \left|e(t)\right| dt \\
&\leq |f_{M_1}|M_1'\omega^{\frac{1}{2}} + \|a\|M_1\omega + \frac{\|b\|\omega}{M_2^\alpha} + \|e\|\omega := M_3,
\end{aligned}
\tag{2.11}
$$

where $|f_{M_1}| := \max_{M_2\leq x\leq M_1} |f(x)|$.

Having in mind Eqs. (2.9), (2.10) and (2.11), we define

$$
\Omega := \left\{x \in X : E_1 < x(t) < E_2 \text{ and } \left\|x'\right\| < E_3 \forall t \in \mathbb{R}\right\},
$$

where $X := \{x \in C(\mathbb{R},\mathbb{R}) : x(0) \equiv x(\omega), \forall t \in \mathbb{R}\}$, $0 < E_1 < M_2$, $E_2 > M_1$ and $E_3 > M_3$. Then the conditions (i) and (ii) of Lemma 2.2 are satisfied. From Eqs. (2.4), (2.9) and (2.10), we have

$$
\int_0^\omega \left(a(t)E_1 - \frac{b(t)}{E_1^\alpha}\right) dt \cdot \int_0^\omega \left(a(t)E_2 - \frac{b(t)}{E_2^\alpha}\right) dt < 0.
$$

Therefore, applying 2.2, we see that Eq. (1.1) has at least one positive periodic solution x with

$$x\in\left((\sigma_*)^{\frac{1}{1+\alpha}}-\frac{\omega^{\frac{1}{2}}}{2}\left(\frac{\pi^2\omega(\|a\|(\sigma^*)^{\frac{2}{1+\alpha}}+\|e\|(\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2-\|a\|\omega^2}\right)^{\frac{1}{2}}-\frac{\pi\omega^2(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}}+\|e\|)}{2(\pi^2-\|a\|\omega^2)},\right.$$
$$\left.(\sigma^*)^{\frac{1}{1+\alpha}}+\frac{\omega^{\frac{1}{2}}}{2}\left(\frac{\pi^2\omega(\|a\|(\sigma^*)^{\frac{2}{1+\alpha}}+\|e\|(\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2-\|a\|\omega^2}\right)^{\frac{1}{2}}+\frac{\pi\omega^2(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}}+\|e\|)}{2(\pi^2-\|a\|\omega^2)}\right).$$

□

Next, we address the condition on the existence of a periodic solution for Eq. (1.1) with strong singularity.

Proof of Theorem 1.2 Similar to the proof of Theorem 1.1, from (2.4) and condition (H_4), we know that there exists a point $\zeta\in(0,\omega)$ such that

$$(\eta_*)^{\frac{1}{1+\alpha}}\le x(\zeta)\le\left(\eta^*\right)^{\frac{1}{1+\alpha}},\tag{2.12}$$

where $\eta^*:=\frac{b}{\min_{t\in[0,\omega]}a(t)}$, $\eta_*:=\frac{b}{\max_{t\in[0,\omega]}a(t)}$. From Eq. (2.9), we get

$$x(t)\le M_1.$$

Next, we claim that there exist two positive constants M_2^* and M_3^* such that

$$x(t)\ge M_2^*,\quad\text{and}\quad\|x'\|\le\lambda M_3^*,\quad\lambda\in(0,1).$$

In fact, we first consider $\int_0^\omega|\frac{b}{x^\alpha(t)}|\,dt$ from condition (H_4). Since $b>0$ and $x(t)>0$, from Eqs. (2.3) and (2.9), we obtain

$$\begin{aligned}\int_0^\omega\left|\frac{b}{x^\alpha(t)}\right|dt&=\int_0^\omega\frac{b}{x^\alpha(t)}\,dt\\&=\int_0^\omega a(t)x(t)\,dt\\&\le\|a\|M_1\omega.\end{aligned}\tag{2.13}$$

Afterwards, from Eqs. (2.9), (2.11) and (2.13), we have

$$\begin{aligned}\|x'\|&\le\lambda\left(\int_0^\omega\left|f\left(x(t)\right)\right|\left|x'(t)\right|dt+\|a\|\int_0^\omega\left|x(t)\right|dt+\int_0^\omega\left|\frac{b}{x^\alpha(t)}\right|dt+\int_0^\omega\left|e(t)\right|dt\right)\\&\le\lambda\left(|f_{M_1}|M_1'\omega^{\frac{1}{2}}+2\|a\|M_1\omega+\|e\|\omega\right):=\lambda M_3^*.\end{aligned}\tag{2.14}$$

On the other hand, multiplying both sides of Eq. (2.2) by $x'(t)$ and integrating on $[\zeta,t]$, where $x(\zeta)\ge(\sigma_*)^{\frac{1}{1+\alpha}}$ is as in Eq. (2.12), we see that

$$\begin{aligned}&\int_\zeta^t x''(s)x'(s)\,ds+\lambda\int_\zeta^t f\left(x(s)\right)\left|x'(s)\right|^2ds+\lambda\int_\zeta^t a(s)x(s)x'(s)\,ds\\&=\lambda\int_\zeta^t\frac{bx'(s)}{x^\alpha(s)}\,ds+\lambda\int_\zeta^t e(s)x'(s)\,ds.\end{aligned}$$

Furthermore, from Eq. (2.9) and (2.14), it is clear that

$$\begin{aligned}
\lambda b\left|\int_{x(\zeta)}^{x(t)}\frac{dv}{v^{\alpha}}\right| &= \lambda\left|\int_{\zeta}^{t}\frac{bx'(s)}{x^{\alpha}(s)}\,ds\right| \\
&= \left|\int_{\zeta}^{t}x''(s)x'(s)\,ds+\lambda\int_{\zeta}^{t}f\big(x(s)\big)\big|x'(s)\big|^{2}\,ds\right. \\
&\quad\left.+\lambda\int_{\zeta}^{t}a(s)x(s)x'(s)\,ds-\lambda\int_{\zeta}^{t}e(s)x'(s)\,ds\right| \\
&\le\frac{1}{2}\big(x'^{2}(t)-x'^{2}(\zeta)\big)+\lambda\int_{0}^{\omega}\big|f\big(x(s)\big)\big|\big|x'(s)\big|^{2}\,ds \\
&\quad+\lambda\int_{0}^{\omega}\big|a(s)\big|\big|x'(s)\big|^{2}\,ds+\lambda\int_{0}^{\omega}\big|e(s)\big|\big|x'(s)\big|\,ds \\
&\le\lambda^{2}\big(M_{3}^{*}\big)^{2}+\lambda|f_{M_{1}}|M_{3}^{*}\omega+\lambda\|a\|M_{1}M_{3}^{*}\omega+\lambda\|e\|M_{3}^{*}\omega,
\end{aligned}$$

since $b > 0$. Therefore, the above inequality implies

$$\left|\int_{x(\zeta)}^{x(t)}\frac{dv}{v^{\alpha}}\right|\le\frac{M_{3}^{*}}{b}\big(M_{3}^{*}+|f_{M_{1}}|M_{3}^{*}\omega+\|a\|M_{1}\omega+\|e\|\omega\big):=M_{2}'. \tag{2.15}$$

Since $\alpha\ge 1$, we get

$$\left|\lim_{x\to 0^{+}}\int_{x}^{1}\frac{dv}{v^{\alpha}}\right|=\frac{1}{1-\alpha}+\frac{1}{\alpha-1}\lim_{x\to 0^{+}}\frac{1}{x^{\alpha-1}}=+\infty. \tag{2.16}$$

From Eq. (2.16) and $x(\zeta)\ge(\eta_{*})^{\frac{1}{1+\alpha}}$, there exists a constant $M_{2}^{*}\in(0,(\eta_{*})^{\frac{1}{1+\alpha}})$ such that

$$\int_{M_{2}^{*}}^{(\eta_{*})^{\frac{1}{1+\alpha}}}\frac{dv}{v^{\alpha}}>M_{2}'. \tag{2.17}$$

Thus, if there is a point $\zeta_{1}\in[\zeta,t]$ such that $x(\zeta_{1})\le M_{2}^{*}$, then

$$\left|\int_{x(\zeta_{1})}^{x(\zeta)}\frac{dv}{v^{\alpha}}\right|\ge\left|\int_{M_{2}^{*}}^{(\eta_{*})^{\frac{1}{1+\alpha}}}\frac{dv}{v^{\alpha}}\right|>M_{2}', \tag{2.18}$$

which contradicts (2.15). Therefore, we get $x(t)>M_{2}^{*}$ for all $t\in[\zeta,\omega]$. For the case $t\in[0,\zeta]$, we can proceed similarly.

The proof is the same as Theorem 1.1. □

Remark 2.2 It is worth mentioning that the method of Theorem 1.2 is no longer applicable to the proof of existence of a positive periodic solution for Eq. (1.1) with weak singularity (i.e. $0<\alpha<1$). Due to $0<\alpha<1$, we cannot get the result that Eq. (2.16) holds, so we do not deduce that Eqs. (2.15) and (2.18) are a contradiction.

Remark 2.3 In condition (H_4), we require that $b(t)\equiv b$. Actually, if $b(t)$ is a continuous periodic function rather than a positive constant, the work on estimating a lower bound of a positive periodic solution for Eq. (1.1) is no longer applicable. In fact, due to $b(t)$

being a function and $b(t) \not\equiv$constant, it is easy to verify that $|\int_{\zeta}^{t} \frac{b(s)x'(s)}{x^{\alpha}(s)}\,ds| \neq |b||\int_{x(\zeta)}^{x(t)} \frac{dv}{v^{\alpha}}\,ds|$. Therefore, we cannot get Eq. (2.15).

Finally, we illustrate our results with two numerical examples.

Example 2.1 Consider the following Liénard equation with weak singularity:

$$x'' + x^3 x' + \left(\frac{1}{4}\sin 8\pi t + \frac{1}{2}\right)x = \frac{\sin^2 4\pi t + 6}{x^{\frac{1}{2}}} + \cos 8\pi t. \tag{2.19}$$

It is clear that $T = \frac{1}{4}$, $\alpha = \frac{1}{2}$, $f(x) = x^3$, $a(t) = \frac{1}{4}\sin 8\pi t + \frac{1}{2}$, $b(t) = \sin^2 4\pi t + 6$, $e(t) = \cos 8\pi t$, $\sigma_* = 8$, $\sigma^* = 28$, and $\|a\| = \frac{3}{4} < 16\pi^2$, conditions (H_1) and (H_2) are satisfied. Next, we consider condition (H_3),

$$\begin{aligned}
&\frac{\omega^{\frac{1}{2}}}{2}\left(\frac{\pi^2\omega(\|a\|(\sigma^*)^{\frac{2}{1+\alpha}} + \|e\|(\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2 - \|a\|\omega^2}\right)^{\frac{1}{2}} + \frac{\pi\omega^2(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}} + \|e\|)}{2(\pi^2 - \|a\|\omega^2)} \\
&\quad = \frac{1}{4} \times \left(\frac{\pi^2 \times \frac{1}{4} \times (\frac{3}{4} \times (28)^{\frac{4}{3}} + 1 \times (28)^{\frac{2}{3}})}{\pi^2 - \frac{3}{64}}\right)^{\frac{1}{2}} + \frac{\pi \times \frac{1}{4} \times (\frac{3}{2} \times (28)^{\frac{2}{3}} + 1)}{2\pi^2 - \frac{3}{32}} \\
&\quad \approx \frac{1}{4} \times (18.3355)^{\frac{1}{2}} + 0.594 \\
&\quad \approx 1.6668 < 4 = (\sigma_*)^{\frac{1}{1+\alpha}}.
\end{aligned}$$

Furthermore, we get

$$\begin{aligned}
&(\sigma_*)^{\frac{1}{1+\alpha}} - \frac{\omega^{\frac{1}{2}}}{2}\left(\frac{\pi^2\omega(\|a\|(\sigma^*)^{\frac{2}{1+\alpha}} + \|e\|(\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2 - \|a\|\omega^2}\right)^{\frac{1}{2}} - \frac{\pi\omega^2(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}} + \|e\|)}{2(\pi^2 - \|a\|\omega^2)} \\
&\quad > 4 - 1.6668 = 2.3332, \\
&(\sigma^*)^{\frac{1}{1+\alpha}} + \frac{\omega^{\frac{1}{2}}}{2}\left(\frac{\pi^2\omega(\|a\|(\sigma^*)^{\frac{2}{1+\alpha}} + \|e\|(\sigma^*)^{\frac{1}{1+\alpha}})}{\pi^2 - \|a\|\omega^2}\right)^{\frac{1}{2}} + \frac{\pi\omega^2(2\|a\|(\sigma^*)^{\frac{1}{1+\alpha}} + \|e\|)}{2(\pi^2 - \|a\|\omega^2)} \\
&\quad < 9.221 + 1.6668 = 10.8878.
\end{aligned}$$

Therefore, applying Theorem 1.1, we know that Eq. (2.19) has at least one positive $\frac{1}{4}$-periodic solution x with $x \in (2.3332, 10.8878)$. Moreover, using Matlab, we can find a positive periodic solution for this equation as shown in Fig. 1.

Example 2.2 Consider the following Liénard equation with strong singularity:

$$x'' + x^2 x' + \left(\sin^2 2t + \frac{1}{2}\right)x = \frac{5}{x^4}. \tag{2.20}$$

It is obvious that $T = \frac{\pi}{2}$, $a(t) = \sin^2 2t + \frac{1}{2}$, $b = 5$ $\|a\| = \frac{3}{2} < 4$, $\alpha = 4$, then conditions (H_1), (H_2) and (H_4) are satisfied. Hence, applying Theorem 1.2, we see that Eq. (2.20) has at least one positive $\frac{\pi}{2}$-periodic solution. Moreover, using Matlab, we can find a positive periodic solution for this equation as shown in in Fig. 2.

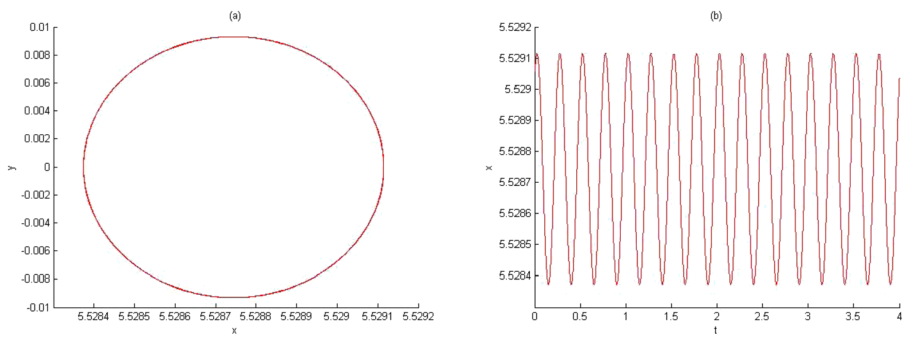

Figure 1 The first picture shows the system response in the (x, y). The second shows the system response in the (t, x) over the time interval of $0 \leq t \leq 50$. The initial conditions are $x_0 = 5.52907$, $y_0 = 0$, $t_0 = 0$

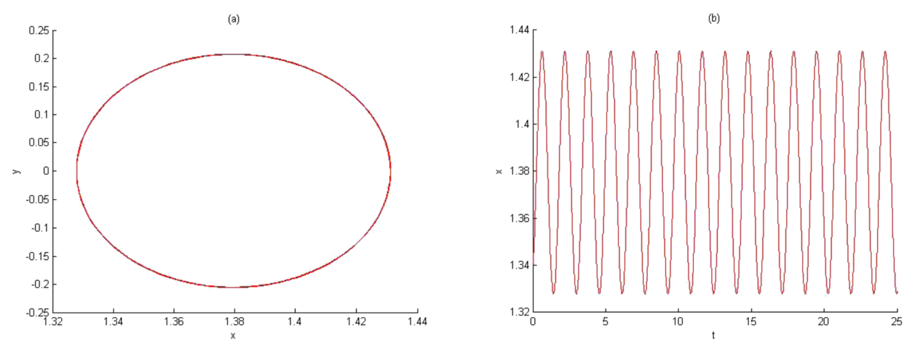

Figure 2 The first picture shows the system response in the (x, y). The second shows the system response in the (t, x) over the time interval of $0 \leq t \leq 50$. The initial conditions are $x_0 = 2$, $y_0 = 0$, $t_0 = 0$

3 Conclusions

In this paper, applying an extension of the Manásevich–Mawhin continuation theorem, we investigate the existence of a periodic solution for Eq. (1.1), where the external force $e(t)$ may change sign, the singular term $\frac{b(t)}{x^\alpha}$ satisfies weak and strong singularities of repulsive type. Besides, we give the existence interval of periodic solution of Eq. (1.1). At last, two examples and numerical solutions (phase portraits and time portraits of periodic solutions of the example) are given to show applications of the theorem. The techniques used of this paper are quite different from that in [5, 8, 10, 17] and our results are more general than those in [5, 8, 10, 17] in two aspects. We first obtain the existence of a positive periodic solution for Eq. (1.1) with weak and strong singularities if the external force $e(t)$ may change sign. Secondly, we give the existence interval of positive periodic solution of Eq. (1.1).

Acknowledgements
YX and GXH are grateful to anonymous referees for their constructive comments and suggestions, which have greatly improved this paper.

Funding
This work was supported by Education Department of Henan Province project (No. 16B110006), Fundamental Research Funds for the Universities of Henan Province (NSFRF170302).

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
YX and GXH contributed to each part of this study equally and declare that they have no competing interests.

Competing interests
YX and GXH declare that they have no competing interests.

Consent for publication
YX and GXH read and approved the final version of the manuscript.

Authors' contributions
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Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 6 June 2020 Accepted: 14 August 2020 Published online: 28 August 2020

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