Kim et al. *Advances in Difference Equations* (2020) 2020:311
https://doi.org/10.1186/s13662-020-02778-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Some results on degenerate Daehee and Bernoulli numbers and polynomials

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Abstract

In this paper, we study a degenerate version of the Daehee polynomials and numbers, namely the degenerate Daehee polynomials and numbers, which were actually called the degenerate Daehee polynomials and numbers of the third kind and recently introduced by Jang et al. (J. Comput. Appl. Math. 364:112343, 2020). We derive their explicit expressions and some identities involving them. Further, we introduce the multiple degenerate Daehee numbers and higher-order degenerate Daehee polynomials and numbers which can be represented in terms of integrals on the unitcube. Again, we deduce their explicit expressions and some identities related to them.

MSC: 11B83; 11B73; 05A19

Keywords: Degenerate Daehee polynomials and numbers; Multiple degenerate Daehee numbers; Higher-order degenerate Daehee polynomials and numbers

1 Introduction

The degenerate versions of Bernoulli and Euler polynomials, namely the degenerate Bernoulli and Euler polynomials, were studied by Carlitz in [1]. In recent years, studying various degenerate versions of some special polynomials and numbers drew attention of some mathematicians and many arithmetic and combinatorial results were obtained [4, 5, 8, 12, 13, 15–17, 19, 20]. They can be explored by using various tools like combinatorial methods, generating functions, differential equations, umbral calculus techniques, p-adic analysis, and probability theory.

The aim of this paper is to study a degenerate version of the Daehee polynomials and numbers, namely the degenerate Daehee polynomials and numbers, in the spirit of [1]. They were actually called the degenerate Daehee polynomials and numbers of the third kind and recently introduced by Jang et al. in [4]. We derive their explicit expressions and some identities involving them. Further, we introduce the multiple degenerate Daehee numbers and higher-order degenerate Daehee polynomials and numbers. Again, we deduce their explicit expressions and some identities related to them.

This paper is organized as follows. In Sect. 1, we state what we need in the rest of the paper. These include the Stirling numbers of the first and second kinds, the higher-order Bernoulli polynomials, the higher-order Daehee polynomials, the higher-order degenerate Bernoulli polynomials, the degenerate exponential functions, and the degenerate Stirling numbers of the first and second kinds. In Sect. 2, we recall the degenerate Daehee polynomials and numbers (of the third kind) from [4] whose generating functions can be expressed in terms of integrals on the unit interval. We find their explicit expressions and some identities involving them. We also introduce the multiple degenerate Daehee numbers, the generating function of which can be expressed in terms of a multiple integral on the unitcube or of the modified polyexponential function [7]. We deduce an explicit expression of them and some identities involving them. In Sect. 3, we introduce the higher-order degenerate Daehee polynomials and numbers whose generating function can be represented as a multiple integral on the unitcube. We derive their explicit expressions and some identities relating to them. Finally, we conclude this paper in Sect. 4.

For $n \geq 0$, the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \quad \bigl(\text{see [8, 14]}\bigr), \tag{1.1}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

As an inversion formula of (1.1), the Stirling numbers of the second kind are defined as

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, (n \geq 0) \quad \bigl(\text{see [8, 14, 16]}\bigr). \tag{1.2}$$

For $\alpha \in \mathbb{N}$, the Bernoulli polynomials of order α are defined as

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \quad \bigl(\text{see [4, 16]}\bigr). \tag{1.3}$$

$B_n(x) = B_n^{(1)}(x)$ are called the Bernoulli polynomials and $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ the Bernoulli numbers of order α.

The Daehee polynomials are defined by

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!} \quad \bigl(\text{see [2, 3, 6, 9–11, 14, 15, 18, 21–34]}\bigr). \tag{1.4}$$

For $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers.

Recently, Jang–Kim–Kwon–Kim studied some new results on degenerate Daehee polynomials and numbers of the third kind (see [4]).

That is, they derived a new integral representation for the degenerate Daehee number and polynomials, the higher-order λ-Daehee numbers and polynomials, and the higher-order twisted λ-Daehee numbers and polynomials (see [4]).

The Daehee polynomials of order k are defined by

$$\left(\frac{\log(1+t)}{t}\right)^{k}(1+t)^x = \sum_{n=0}^{\infty} D_n^{(k)}(x)\frac{t^n}{n!} \quad \bigl(\text{see [4, 10]}\bigr). \tag{1.5}$$

In [9], we note that

$$D_m^{(k)}(z) = m! \sum_{n=0}^{m} \binom{z}{m-n} b_m^{(-k)}, \tag{1.6}$$

where $b_n^{(x)}$ are the higher-order Bernoulli numbers of the second kind given by

$$\left(\frac{t}{\log(1+t)}\right)^x = \sum_{n=0}^{\infty} b_n^{(x)} t^n. \tag{1.7}$$

Recently, Daehee numbers and polynomials have been studied by many researchers in various areas (see [2, 3, 6, 9–11, 14, 15, 18, 21–34]).

In [1], Carlitz considered the degenerate Bernoulli polynomials given by

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(x)} \frac{t^n}{n!} \quad (\lambda \in \mathbb{R}). \tag{1.8}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers. For $r \in \mathbb{N}$, he also defined the higher–order degenerate Bernoulli polynomials as

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x) \frac{t^n}{n!} \quad \bigl(\text{see [16]}\bigr). \tag{1.9}$$

When $x = 0$, $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$ are called the degenerate Bernoulli numbers of order r.

The degenerate exponential functions are given by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}} \quad \bigl(\text{see [7–9, 12–14, 16–20]}\bigr). \tag{1.10}$$

We note that

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda}}{n!} t^n \quad \bigl(\text{see [8]}\bigr), \tag{1.11}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$ $(n \geq 1)$.

Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$, $\lim_{\lambda \to 0} \beta_{n,\lambda}^{(r)}(x) = B_n^{(r)}(x)$.

Recently, Kim considered the degenerate Stirling numbers of the second kind given by

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l, (n \geq 0) \quad \bigl(\text{see [8]}\bigr). \tag{1.12}$$

Note that $\lim_{\lambda\to 0} S_{2,\lambda}(n,l) = S_2(n,l)$.

From (1.12), we note that

$$\frac{1}{k!}\bigl(e_\lambda(t) - 1\bigr)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!} \quad (k \geq 0)\bigl(\text{see [8]}\bigr). \tag{1.13}$$

As an inversion formula of (1.12), the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda} \quad (n \geq 0)\big(\text{see [8]}\big). \tag{1.14}$$

We see that $\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1)$ is the compositional inverse of $e_\lambda(t)$ satisfying $\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t$.

By (1.14), we get

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \quad \big(\text{see [8]}\big). \tag{1.15}$$

Note that $\lim_{\lambda\to 0}\log_\lambda(1+t) = \log(1+t)$.

2 Degenerate Daehee numbers and polynomials

The degenerate Daehee polynomials are defined by (see [4])

$$\frac{\log_\lambda(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!}, \quad (\lambda \in \mathbb{R}). \tag{2.1}$$

When $x = 0$, $D_{n,\lambda} = D_{n,\lambda}(0)$ are called the degenerate Daehee numbers.

From (1.4) and (2.1), we note that $\lim_{\lambda\to 0} D_{n,\lambda}(x) = D_n(x)$ $(n \geq 0)$.

We observe that

$$\frac{\log(1+t)}{t}\int_0^1 (1+t)^{\lambda y + x}\, dy = \frac{\log_\lambda(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!}.$$

When $x = 0$, we have

$$\frac{\log(1+t)}{t}\int_0^1 (1+t)^{\lambda y}\, dy = \sum_{n=0}^{\infty} D_{n,\lambda}\frac{t^n}{n!}. \tag{2.2}$$

On the other hand,

$$\begin{aligned}
&\frac{\log(1+t)}{t}\int_0^1 (1+t)^{\lambda y}\, dy \\
&= \frac{\log(1+t)}{t}\sum_{m=0}^{\infty}\frac{\lambda^m(\log(1+t))^m}{(m+1)!} \\
&= \frac{1}{t}\sum_{m=0}^{\infty}\frac{(\log(1+t))^{m+1}}{(m+1)!}\lambda^m = \frac{1}{t}\sum_{m=1}^{\infty}\lambda^{m-1}\frac{1}{m!}\big(\log(1+t)\big)^m \\
&= \frac{1}{t}\sum_{m=1}^{\infty}\lambda^{m-1}\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!} = \frac{1}{t}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\lambda^{m-1}S_1(n,m)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{m=1}^{n+1}\lambda^{m-1}S_1(n+1,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$, we have*

$$D_{n,\lambda} = \frac{1}{n+1} \sum_{m=1}^{n+1} \lambda^{m-1} S_1(n+1,m).$$

By replacing t by $e_\lambda(t) - 1$ in (2.1), we get

$$\begin{aligned}
\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) &= \sum_{m=0}^{\infty} D_{m,\lambda}(x) \frac{1}{m!} \bigl(e_\lambda(t) - 1\bigr)^m \\
&= \sum_{m=0}^{\infty} D_{m,\lambda}(x) \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} D_{m,\lambda}(x) S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.4}$$

On the other hand,

$$\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n}. \tag{2.5}$$

Therefore, by (2.4) and (2.5), we obtain the following theorem.

Theorem 2.2 *For $n \geq 0$, we have*

$$\beta_{n,\lambda}(x) = \sum_{m=0}^{n} D_{m,\lambda}(x) S_{2,\lambda}(n,m).$$

Note that

$$B_n(x) = \lim_{\lambda \to 0} \beta_{n,\lambda}(x) = \sum_{m=0}^{n} D_m(x) S_2(n,m) \quad (n \geq 0).$$

To find the inversion formula of Theorem 2.2, we replace t by $\log_\lambda(1+t)$ in (1.8) and get

$$\begin{aligned}
\frac{\log_\lambda(1+t)}{t} (1+t)^x &= \sum_{m=0}^{\infty} \beta_{m,\lambda}(x) \frac{1}{m!} \bigl(\log_\lambda(1+t)\bigr)^m \\
&= \sum_{m=0}^{\infty} \beta_{m,\lambda}(x) \sum_{n=m}^{\infty} S_{1,\lambda}(n,m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \beta_{m,\lambda}(x) S_{1,\lambda}(n,m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.6}$$

Therefore, by (2.1) and (2.6), we obtain the following theorem.

Theorem 2.3 *For $n \geq 0$, we have*

$$D_{n,\lambda}(x) = \sum_{m=0}^{n} \beta_{m,\lambda}(x) S_{1,\lambda}(n,m).$$

Note that

$$D_n(x) = \lim_{\lambda\to 0} D_{n,\lambda}(x) = \sum_{m=0}^{n} B_m(x)S_1(n,m) \quad (n \geq 0).$$

From (1.10), we can derive the following equation:

$$\begin{aligned}
\sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{\log_\lambda(1+t)}{t}(1+t)^x = \frac{\log_\lambda(1+t)}{t}e_\lambda^x\big(\log_\lambda(1+t)\big) \\
&= \frac{\log_\lambda(1+t)}{t}\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{(\log_\lambda(1+t))^m}{m!} \\
&= \frac{1}{t}\sum_{m=0}^{\infty}(m+1)(x)_{m,\lambda}\frac{1}{(m+1)!}\big(\log_\lambda(1+t)\big)^{m+1} \\
&= \frac{1}{t}\sum_{m=0}^{\infty}(m+1)(x)_{m,\lambda}\sum_{n=m+1}^{\infty}S_{1,\lambda}(n,m+1)\frac{t^n}{n!} \qquad (2.7)\\
&= \sum_{m=0}^{\infty}(m+1)(x)_{m,\lambda}\sum_{n=m}^{\infty}\frac{S_{1,\lambda}(n+1,m+1)}{n+1}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left\{\frac{1}{n+1}\sum_{m=0}^{n}(m+1)(x)_{m,\lambda}S_{1,\lambda}(n+1,m+1)\right\}\frac{t^n}{n!}. \qquad (2.8)
\end{aligned}$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$, we have*

$$D_{n,\lambda}(x) = \frac{1}{n+1}\sum_{m=0}^{n}(m+1)(x)_{m,\lambda}S_{1,\lambda}(n+1,m+1).$$

For $s \in \mathbb{C}$, the polyexponential function is defined by Hardy as

$$e(x,a|s) = \sum_{n=0}^{\infty}\frac{x^n}{(n+a)^s n!}, \big(\mathrm{Re}(a) > 0\big) \quad \big(\text{see [16]}\big). \qquad (2.9)$$

In [7], the modified polyexponential function is introduced as

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!n^k} \quad (k \in \mathbb{Z}). \qquad (2.10)$$

Note that $x\, e(x,1|k) = \mathrm{Ei}_k(x)$.

We observe that

$$\frac{\partial}{\partial x_1}(1+t)^{\lambda x_1 x_2\cdots x_k} = x_2\cdots x_k\lambda\log(1+t)(1+t)^{\lambda x_1 x_2\cdots x_k}. \qquad (2.11)$$

Thus, by (2.11), we get

$$\begin{aligned} &\frac{\log(1+t)}{t}\int_0^1 (1+t)^{\lambda x_1x_2\cdots x_k}\,dx_1 \\ &= \frac{1}{x_2x_3\cdots x_k}\frac{\log_\lambda(1+t)^{x_2\cdots x_k}}{t} \\ &= \frac{1}{t}\frac{1}{x_2x_3\cdots x_k}\sum_{m=1}^\infty \lambda^{m-1}\frac{(\log(1+t))^m}{m!}x_2^m x_3^m\cdots x_k^m \\ &= \frac{1}{t}\sum_{m=1}^\infty \frac{\lambda^{m-1}(\log(1+t))^m}{(m-1)!m}x_2^{m-1}x_3^{m-1}\cdots x_k^{m-1}. \end{aligned} \tag{2.12}$$

From (2.12), we can derive the following equation:

$$\begin{aligned} &\frac{\log(1+t)}{t}\int_0^1\cdots\int_0^1 (1+t)^{\lambda x_1x_2\cdots x_k}\,dx_1\,dx_2\cdots dx_k \\ &= \frac{1}{t}\sum_{m=1}^\infty \frac{\lambda^{m-1}(\log(1+t))^m}{(m-1)!m^k} = \frac{1}{\lambda t}\mathrm{Ei}_k\bigl(\lambda\log(1+t)\bigr). \end{aligned} \tag{2.13}$$

Now, we define the multiple degenerate Daehee numbers as the multiple integral on the unitcube given by

$$\frac{\log(1+t)}{t}\int_0^1\cdots\int_0^1 (1+t)^{\lambda x_1x_2\cdots x_k}\,dx_1\,dx_2\cdots dx_k = \sum_{n=0}^\infty \widehat{D}_{n,\lambda}^{(k)}\frac{t^n}{n!}. \tag{2.14}$$

Then, by (2.13) and (2.14), we get

$$\frac{1}{\lambda t}\mathrm{Ei}_k\bigl(\lambda\log(1+t)\bigr) = \sum_{n=0}^\infty \widehat{D}_{n,\lambda}^{(k)}\frac{t^n}{n!}. \tag{2.15}$$

Note that $\widehat{D}_{n,\lambda}^{(1)} = D_{n,\lambda}$ $(n \geq 0)$.

We observe that

$$\begin{aligned} \frac{1}{\lambda t}\mathrm{Ei}_k\bigl(\lambda\log(1+t)\bigr) &= \frac{1}{\lambda t}\sum_{m=1}^\infty \frac{\lambda^m(\log(1+t))^m}{(m-1)!m^k} \\ &= \frac{1}{\lambda t}\sum_{m=1}^\infty \frac{\lambda^m}{m^{k-1}}\frac{1}{m!}\bigl(\log(1+t)\bigr)^m \\ &= \frac{1}{\lambda t}\sum_{m=1}^\infty \frac{\lambda^m}{m^{k-1}}\sum_{n=m}^\infty S_1(n,m)\frac{t^n}{n!} \\ &= \frac{1}{t}\sum_{n=1}^\infty\sum_{m=1}^n \frac{\lambda^{m-1}}{m^{k-1}}S_1(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^\infty\left(\frac{1}{n+1}\sum_{m=1}^{n+1}\frac{\lambda^{m-1}}{m^{k-1}}S_1(n+1,m)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.16}$$

Therefore, by (2.15) and (2.16), we obtain the following theorem.

Theorem 2.5 *For $n \geq 0$, we have*

$$\widehat{D}_{n,\lambda}^{(k)} = \frac{1}{n+1}\sum_{m=1}^{n+1}\frac{\lambda^{m-1}}{m^{k-1}}S_1(n+1,m).$$

By replacing t by $e^t - 1$ in (2.15), we get

$$\begin{aligned}\sum_{m=0}^{\infty}\widehat{D}_{m,\lambda}^{(k)}\frac{1}{m!}\left(e^t-1\right)^m &= \frac{1}{\lambda(e^t-1)}\mathrm{Ei}_k(\lambda t)\\ &= \frac{t}{e^t-1}\frac{1}{\lambda t}\mathrm{Ei}_k(\lambda t) = \sum_{l=0}^{\infty}B_l\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{\lambda^m}{(m+1)^k}\frac{t^m}{m!}.\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{\lambda^{n-l}B_l}{(n-l+1)^k}\right)\frac{t^n}{n!}. \qquad (2.17)\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{m=0}^{\infty}\widehat{D}_{m,\lambda}^{(k)}\frac{1}{m!}\left(e^t-1\right)^m &= \sum_{m=0}^{\infty}\widehat{D}_{m,\lambda}^{(k)}\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{D}_{m,\lambda}^{(k)}S_2(n,m)\right)\frac{t^n}{n!}. \qquad (2.18)\end{aligned}$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 2.6 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}\widehat{D}_{m,\lambda}^{(k)}S_2(n,m) = \sum_{l=0}^{n}\binom{n}{l}\frac{\lambda^{n-l}B_l}{(n-l+1)^k}.$$

From (2.17), we note that

$$\begin{aligned}\frac{1}{\lambda t}\mathrm{Ei}_k(\lambda t) &= \frac{1}{t}\left(e^t-1\right)\sum_{m=0}^{\infty}\widehat{D}_{m,\lambda}^{(k)}\frac{1}{m!}\left(e^t-1\right)^m\\ &= \frac{1}{t}\sum_{m=1}^{\infty}m\widehat{D}_{m-1,\lambda}^{(k)}\frac{1}{m!}\left(e^t-1\right)^m\\ &= \frac{1}{t}\sum_{m=1}^{\infty}m\widehat{D}_{m-1,\lambda}^{(k)}\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!}\\ &= \frac{1}{t}\sum_{n=1}^{\infty}\sum_{m=1}^{n}m\widehat{D}_{m-1,\lambda}^{(k)}S_2(n,m)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{m=1}^{n+1}m\widehat{D}_{m-1,\lambda}^{(k)}S_2(n+1,m)\right)\frac{t^n}{n!}. \qquad (2.19)\end{aligned}$$

On the other hand,

$$\frac{1}{\lambda t}\mathrm{Ei}_k(\lambda t)=\frac{1}{\lambda t}\sum_{n=1}^{\infty}\frac{\lambda^n t^n}{(n-1)!n^k}=\sum_{n=0}^{\infty}\frac{\lambda^n}{(n+1)^k}\frac{t^n}{n!}. \tag{2.20}$$

Therefore, by (2.19) and (2.20), we obtain the following theorem.

Theorem 2.7 *For $n\geq 0$, we have*

$$\begin{aligned}\frac{\lambda^n}{(n+1)^k}&=\frac{1}{n+1}\sum_{m=1}^{n}m\widehat{D}_{m-1,\lambda}^{(k)}S_2(n+1,m)\\&=\frac{1}{n+1}\sum_{m=0}^{n-1}(m+1)\widehat{D}_{m,\lambda}^{(k)}S_2(n+1,m+1).\end{aligned}$$

3 Higher-order degenerate Daehee numbers and polynomials

As an additive version of (2.14), we consider the degenerate Daehee polynomials of order r given by the following multiple integral on the unit cube:

$$\begin{aligned}\sum_{n=0}^{\infty}D_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}&=\left(\frac{\log(1+t)}{t}\right)^r\int_0^1\cdots\int_0^1(1+t)^{\lambda(x_1+\cdots+x_r)+x}\,dx_1\cdots dx_r\\&=\left(\frac{\log_\lambda(1+t)}{t}\right)^r(1+t)^x\quad(r\in\mathbb{N}).\end{aligned}\tag{3.1}$$

When $x=0$, $D_{n,\lambda}^{(r)}=D_{n,\lambda}^{(r)}(0)$ $(n\geq 0)$, are called the degenerate Daehee numbers of order r.

From (3.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}D_{n,\lambda}^{(r)}\frac{t^n}{n!}&=\left(\frac{\log_\lambda(1+t)}{t}\right)^r=\frac{r!}{t^r}\frac{1}{r!}\bigl(\log_\lambda(1+t)\bigr)^r\\&=\frac{r!}{t^r}\sum_{n=r}^{\infty}S_{1,\lambda}(n,r)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}S_{1,\lambda}(n+r,r)\frac{r!n!}{(n+r)!}\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\frac{S_{1,\lambda}(n+r,r)}{\binom{n+r}{n}}\frac{t^n}{n!}.\end{aligned}\tag{3.2}$$

Therefore, by comparing the coefficients on both sides of (3.2), we obtain the following theorem.

Theorem 3.1 *For $n\geq 0$, we have*

$$D_{n,\lambda}^{(r)}=\frac{1}{\binom{n+r}{n}}S_{1,\lambda}(n+r,r)\quad(r\in\mathbb{N}).$$

By replacing t by $e_\lambda(t)-1$ in (3.1), we get

$$\begin{aligned}\sum_{k=0}^{\infty} D_{k,\lambda}^{(r)}(x)\frac{1}{k!}\big(e_\lambda(t)-1\big)^k &= \left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t)\\ &= \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}.\end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned}&\sum_{k=0}^{\infty} D_{k,\lambda}^{(r)}(x)\frac{1}{k!}\big(e_\lambda(t)-1\big)^k\\ &\quad=\sum_{k=0}^{\infty} D_{k,\lambda}^{(r)}(x)\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}\\ &\quad=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}D_{k,\lambda}^{(r)}(x)S_{2,\lambda}(n,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.4}$$

Therefore, by (3.3) and (3.4), we obtain the following theorem.

Theorem 3.2 *For $n \geq 0$, we have*

$$\beta_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}D_{k,\lambda}^{(r)}(x)S_{2,\lambda}(n,k).$$

By replacing t by $\log_\lambda(1+t)$ in (1.9), we get

$$\begin{aligned}\left(\frac{\log_\lambda(1+t)}{t}\right)^r(1+t)^x &= \sum_{k=0}^{\infty}\beta_{k,\lambda}^{(r)}(x)\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k\\ &= \sum_{k=0}^{\infty}\beta_{k,\lambda}^{(r)}(x)\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\beta_{k,\lambda}^{(r)}(x)S_{1,\lambda}(n,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.5}$$

On the other hand,

$$\left(\frac{\log_\lambda(1+t)}{t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}D_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{3.6}$$

Therefore, by (3.5) and (3.6), we obtain the following theorem.

Theorem 3.3 *For $n \geq 0$, we have*

$$D_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}\beta_{k,\lambda}^{(r)}(x)S_{1,\lambda}(n,k).$$

From (3.1), we note that

$$\sum_{n=0}^{\infty} D_{n,\lambda}^{(r)} \frac{t^n}{n!} = \underbrace{\left(\frac{\log_\lambda(1+t)}{t}\right) \times \cdots \times \frac{\log_\lambda(1+t)}{t}}_{r\text{- times}}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_r=n} \binom{n}{l_1,\ldots l_r} D_{l_1,\lambda} \cdots D_{l_r,\lambda} \right) \frac{t^n}{n!}. \tag{3.7}$$

By (3.7), we get

$$D_{n,\lambda}^{(r)} = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1,\ldots l_r} D_{l_1,\lambda} \cdots D_{l_r,\lambda} \quad (n \geq 0). \tag{3.8}$$

On the other hand, by (3.2), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} D_{n,\lambda}^{(r)} \frac{t^n}{n!} &= \left(\frac{\log(1+t)}{t}\right)^r \int_0^1 \cdots \int_0^1 (1+t)^{\lambda(x_1+\cdots+x_r)}\, dx_1 \cdots dx_r \\
&= \left(\frac{\log(1+t)}{t}\right)^r \sum_{m=0}^{\infty} \lambda^m \frac{(\log(1+t))^m}{m!} \int_0^1 \cdots \int_0^1 (x_1+\cdots+x_r)^m\, dx_1 \cdots dx_r \\
&= \frac{1}{t^r} \sum_{m=0}^{\infty} \lambda^m \sum_{l_1+\cdots+l_r=m} \binom{m}{l_1,\ldots,l_r} \frac{1}{(l_1+1)\cdots(l_r+1)} \frac{(\log(1+t))^{m+r}}{m!} \\
&= \frac{1}{t^r} \sum_{m=0}^{\infty} \lambda^m \sum_{l_1+\cdots+l_r=m} \binom{m}{l_1,\ldots,l_r} \frac{1}{(l_1+1)\cdots(l_r+1)} \frac{(m+r)!}{m!} \\
&\quad \times \sum_{n=m+r}^{\infty} S_1(n,m+r) \frac{t^n}{n!} \\
&= \sum_{m=0}^{\infty} \lambda^m \sum_{l_1+\cdots+l_r=m} \binom{m}{l_1,\ldots,l_r} \frac{1}{(l_1+1)\cdots(l_r+1)} \frac{(m+r)!}{m!} \\
&\quad \times \sum_{n=m}^{\infty} S_1(n+r,m+r) \frac{t^n}{(n+r)!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \lambda^m \sum_{l_1+\cdots+l_r=m} \frac{\binom{n}{l_1,\ldots,l_r}}{(l_1+1)\cdots(1_r+1)} S_1(n+r,m+r) \frac{\binom{m+r}{r}}{\binom{n+r}{r}} \right) \frac{t^n}{n!}.
\end{aligned} \tag{3.9}$$

Therefore, by comparing the coefficients on both sides of (3.9), we obtain the following theorem.

Theorem 3.4 *For $n \geq 0$, we have*

$$\begin{aligned}
D_{n,\lambda}^{(r)} &= \sum_{l_1+\cdots+l_r=m} \binom{n}{l_1,\ldots,l_r} D_{l_1,\lambda} \cdots D_{l_r,\lambda} \\
&= \sum_{m=0}^{n} \lambda^m \sum_{l_1+\cdots+l_r=m} \binom{m}{l_1,\ldots,l_r} \frac{S_1(n+r,m+r)}{(l_1+1)\cdots(1_r+1)} \frac{\binom{m+r}{r}}{\binom{n+r}{r}}.
\end{aligned}$$

4 Conclusion

In the spirit of [1], we studied the degenerate Daehee polynomials and numbers which were actually called the degenerate Daehee polynomials and numbers of the third kind and recently introduced by Jang et al. in [4]. We derived their explicit expressions and some identities involving them. Further, we introduced the multiple degenerate Daehee numbers and higher-order degenerate Daehee polynomials and numbers and deduced their explicit expressions and some identities related to them.

The possible applications of our results are as follows. The first one is their applications to identities of symmetry. For example, in [13] many symmetric identities in three variables, related to degenerate Euler polynomials and alternating generalized falling factorial sums, were obtained. The second one is their applications to differential equations from which we can derive some useful identities. For example, in [12] an infinite family of nonlinear differential equations, having the generating function of the degenerate Bernoulli numbers as a solution, were derived. As a result, an identity, involving the degenerate Bernoulli and higher-order degenerate Bernoulli numbers, were obtained. Similar things had been done for the degenerate Euler numbers. The third one is their applications to probability theory. Indeed, in [19] it was shown that both the degenerate λ-Stirling polynomials of the second and the r-truncated degenerate λ-Stirling polynomials of the second kind appear in certain expressions of the probability distributions of appropriate random variables.

Finally, it is one of our future projects to continue to study various degenerate versions of some special polynomials and their applications to mathematics, science and engineering.

We studied the degenerate Daehee polynomials and numbers which are different from the degenerate Daehee polynomials and numbers of the third kind introduced by Jang et al. [4].

Acknowledgements
We would like to thank the referees for their valuable comments and suggestions. Also, we would like to thank Jangjeon Institute for Mathematical Science for the support of this research.

Funding
This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2017R1E1A1A03070882).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no conflicts of interest.

Authors' contributions
TK and DSK conceived of the framework and structured the whole paper; DSK and TK wrote the paper; JK and HYK checked the results of the paper; DSK and TK completed the revision of the article. All authors have read and agreed to the published version of the manuscript.

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Publisher's Note

Received: 15 May 2020 Accepted: 15 June 2020 Published online: 23 June 2020

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