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Abstract. In this paper, we prove an orbital stability result for the Degasperis-Procesi peakon with respect to perturbations having a momentum density that is first negative and then positive. This leads to the orbital stability of the antipeakon-peakon profile with respect to such perturbations.

1. Introduction

In this paper, we consider the Degasperis-Procesi equation (DP) first derived in [5], usually written as

$$\begin{cases} u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}\,. \end{cases} \tag{1}$$

The DP equation has been proved to be physically relevant for water waves (see [2]) as an asymptotic shallow-water approximation to the Euler equations in some specific regime. It shares a lot of properties with the famous Camassa-Holm equation (CH) that reads

$$u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}\,. \tag{2}$$

In particular, it has a bi-hamiltonian structure, it is completely integrable (see [6]) and has got the same explicit peaked solitary waves. These solitary waves are called *peakons* whenever $c > 0$ and *antipeakons* whenever $c < 0$ and are defined by

$$u(t,x) = \varphi_c(x-ct) = c\varphi(x-ct) = ce^{-|x-ct|}, \quad c \in \mathbb{R}^*, \quad (t,x) \in \mathbb{R}^2\,. \tag{3}$$

Note that to give a sense to these solutions one has to apply $(1-\partial_x^2)^{-1}$ to (1), to rewrite it under the form

$$u_t + \frac{1}{2}\partial_x(u^2) + \frac{3}{2}(1-\partial_x^2)^{-1}\partial_x(u^2) = 0, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{4}$$

However, in contrast with the CH equation, the DP equation has also shock peaked waves (see for instance [14]) which are given by

$$u(t,x) = -\frac{1}{t+k}\operatorname{sgn}(x)e^{-|x|}, \quad k > 0 \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}\,.$$

Another important difference between the CH and the DP equations is due to the fact that the DP conservations laws permit only to control the L^2-norm of the solution whereas the H^1-norm is a conserved quantity for the CH equation. In particular, without any supplementary hypotheses, the solutions of the DP equation may be unbounded contrary to the CH-solutions. In this paper we will make use of the three following conservation laws of the DP equation :

$$M(u) = \int_{\mathbb{R}} y, \quad E(u) = \int_{\mathbb{R}} yv = \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \tag{5}$$

$$\text{and} \qquad F(u) = \int_{\mathbb{R}} u^3 = \int_{\mathbb{R}} \left(-v_{xx}^3 + 12vv_{xx}^2 - 48v^2 v_{xx} + 64v^3\right), \tag{6}$$

where $y = (1-\partial_x^2)u$ and $v = (4-\partial_x^2)^{-1}u$.
It is worth noticing that these two variables, the momentum density $y = (1-\partial_x^2)u$ and the smooth variable

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$v = (4 - \partial_x^2)^{-1} u$ play a crucial role in the DP dynamic. In the sequel we will often make use of the fact that (1) can be rewritten under the form

$$y_t + u y_x + 3 u_x y = 0, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{7}$$

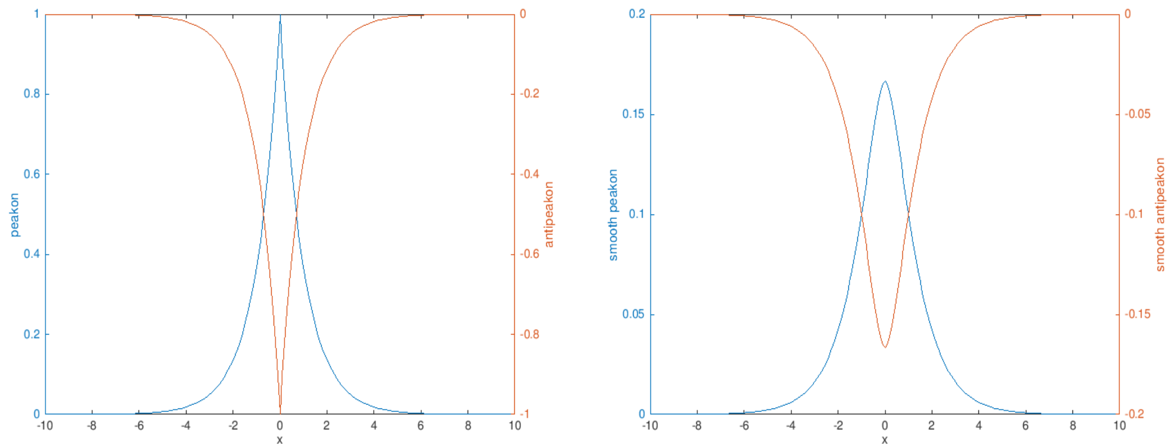
which is a transport equations for the momentum density as well as under the form

$$v_t = -\partial_x (1 - \partial_x^2)^{-1} u^2, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{8}$$

Note that, in the same way as v is associated with u, we will associate with the peakon profile φ_c the so-called *smooth-peakon* profile ρ_c that is given by

$$\rho_c = (4 - \partial_x^2)^{-1} \varphi_c = \frac{1}{4} e^{-2|\cdot|} * \varphi_c = \frac{c}{3} e^{-|\cdot|} - \frac{c}{6} e^{-2|\cdot|} \geq 0 \, . \tag{9}$$

In [13] (see also [10] for a great simplification) an orbital stability[1] result is proven for the DP peakon



(A) Peakon and antipeakon profiles

(B) Smooth peakon and smooth antipeakon profiles

FIGURE 1. (A) Peakon and antipeakon representative curves with speed $c = \pm 1$. They are even functions that admit a single maximum c (resp. maximum $-c$) at the origin. (B) Smooth peakon and smooth antipeakon representative curves with speed $c = \pm 1$. They are even C^2 functions that admit a single maximum $c/6$ (resp. minimum $-c/6$) at the origin.

by adapting the approach first developed by Constantin and Strauss [4] for the Camassa-Holm peakon. However, in deep contrast to the Camassa-Holm case, the proof in [13] (and also in [10]) crucially use that the momentum density of the perturbation is non negative. This is absolutely required for instance in [[13], Lemma 3.5] to get the crucial estimate on the auxiliary function h (see Section 5 for the definition of h)). Up to our knowledge, there is no available stability result for the Degasperis-Procesi peakons without this requirement on the momentum density and one of the main contribution of this work is to give a first stability result for the DP peakon with respect to perturbations that do not share this sign requirement. At this stage, it is worth noticing that the global existence of smooth solutions to the DP equation is only known for initial data that have either a momentum density with a constant sign or a momentum density that is first non negative and then non positive.

The first part of this paper is devoted to the proof of a stability result for the peakon with respect to perturbations that belong to this second class of initial data. We would like to emphasize that the key supplementary argument with respect to the case of a non negative momentum density is of a dynamic nature. Inspired by similar considerations for the Camassa-Holm equation contained in [17], we study the dynamic of the momentum density $y(t)$ at the left of a smooth curve $x(t)$ such that $u(t,\cdot) - \varphi_c(\cdot - x(t))$ remains small for all $t \in [0,T]$ with $T > 0$ large enough. This is in deep contrast with the arguments in the

[1]See [17] for an asymptotic stability result in the class of functions with a positive momentum density.

case $y \geq 0$ and with the common arguments for orbital stability that are of static nature : They only use the conservation laws together with the continuity of the solution.

In a second time, we combine this stability result with some almost monotony results to get the orbital stability of the DP antipeakon-peakon profile and more generally of trains of antipeakon-peakons.

Before stating our results let us introduce some notations and some function spaces that will appear in the statements. For $p \in [1, +\infty]$ we denote by $L^p(\mathbb{R})$ the usual Lebesgue spaces endowed with their usual norm $\|\cdot\|_{L^p}$. We notice that by integration by parts, it holds

$$\|u(t,\cdot)\|_{L^2}^2 = \int_{\mathbb{R}} (4v - v_{xx})^2 dx = \int_{\mathbb{R}} \left(16v^2 + 8v_x^2 + v_{xx}^2\right) dx$$

and thus

$$E(u) \leq \|u\|_{L^2}^2 \leq 4E(u) \; .$$

Therefore, $E(\cdot)$ is equivalent to $\|\cdot\|_{L^2(\mathbb{R})}^2$ and in the sequel of this paper we set

$$\|u\|_{\mathcal{H}} = \sqrt{E(u)} \quad \text{so that} \quad \|u\|_{\mathcal{H}} \leq \|u\|_{L^2} \leq 2\|u\|_{\mathcal{H}} \tag{10}$$

As in [3], we will work in the space Y defined by

$$Y := \Big\{ u \in L^2(\mathbb{R}) \quad \text{with} \qquad u - u_{xx} \in \mathcal{M}(\mathbb{R}) \Big\} \tag{11}$$

where $\mathcal{M}(\mathbb{R})$ is the space of finite Radon measure on $\mathbb{R}$ that is endowed with the norm $\|\cdot\|_{\mathcal{M}}$ where

$$\|y\|_{\mathcal{M}} := \sup_{\varphi \in C(\mathbb{R}), \|\varphi\|_{L^\infty} \leq 1} |\langle y, \varphi \rangle| \; .$$

Hypothesis 1. *We will say that $u_0 \in Y$ satisfies Hypothesis 1 if there exists $x_0 \in \mathbb{R}$ such that its momentum density $y_0 = u_0 - u_{0,xx}$ satisfies*

$$\text{supp } y_0^- \subset]-\infty, x_0] \qquad \text{and} \qquad \text{supp } y_0^+ \subset [x_0, +\infty[. \tag{12}$$

where y_0^+ and y_0^- are respectively the positive and the negative part of the Radon measure y_0.

Theorem 1 (Stability of a single Peakon)**.** *There exists $0 < \varepsilon_0 < 1$ such that for any $c > 0$, $A > 0$ and $0 < \varepsilon < \varepsilon_0 \frac{1 \wedge c^2}{(2+c)^3}$, there exists $0 < \delta = \delta(A, \varepsilon, c) \leq \varepsilon^4$ such that for any $u_0 \in Y$ satisfying Hypothesis 1 with*

$$\|u_0 - \varphi_c\|_{\mathcal{H}} \leq \delta \leq \varepsilon^4 \tag{13}$$

and

$$\|u_0 - u_{0,xx}\|_{\mathcal{M}} \leq A, \tag{14}$$

the emanating solution of the DP equation satisfies

$$\|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{\mathcal{H}} \leq 2(2+c)\, \varepsilon, \;\; \forall t \in \mathbb{R}_+ \tag{15}$$

and

$$\|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{L^\infty} \leq 8(2+c)^2 \varepsilon^{2/3}, \;\; \forall t \in \mathbb{R}_+ \; , \tag{16}$$

where $\xi(t) \in \mathbb{R}$ is the only point where the function $v(t,\cdot) = (4 - \partial_x^2)^{-1} u(t,\cdot)$ reaches its maximum on $\mathbb{R}$.

Combining the above stability of a single peakon with the general framework first introduced in [16] and more precisely following [7]-[8] we obtain the stability of a train of well-ordered antipeakons and peakons. This contains in particular the stability of the antipeakon-peakon profile.

Theorem 2. *Let be given $N_- \in \mathbb{N}^*$ negative velocities $c_{-N_-} < .. < c_{-1} < 0$, $N_+ \in \mathbb{N}^*$ positive velocities $0 < c_1 < .. < c_{N_+}$ and $A > 0$. There exist $B = B(\vec{c}) > 0$, $L_0 = L_0(A, \vec{c}) > 0$ and $0 < \varepsilon_0 = \varepsilon_0(\vec{c}) < 1$ such that for any $0 < \varepsilon < \varepsilon_0(\vec{c})$ there exists $0 < \delta(\varepsilon, A, \vec{c}) < \varepsilon^4$ such that if $u \in C(\mathbb{R}_+; H^1)$ is the solution of the DP equation emanating from $u_0 \in Y$, satisfying Hypothesis 1 with*

$$\|u_0 - u_{0,xx}\|_{\mathcal{M}} \leq A, \tag{17}$$

and

$$\|u_0 - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j^0)\|_{\mathcal{H}} \le \delta \le \varepsilon^4 \tag{18}$$

for some $z^0_{-N_-} < .. < z^0_{-1} < z^0_1 < \cdots < z^0_{N_+}$ *such that*

$$z_j^0 - z_q^0 \ge L \ge L_0, \quad \forall (j,q) \in \Big([[-N_-, N_+]] \setminus \{0\}\Big)^2, \quad j > q \ , \tag{19}$$

then there exist $N_- + N_+$ *functions* $\xi_{-N_-}(\cdot), .., \xi_{-1}(\cdot), \xi_1(\cdot), .., \xi_{N_+}(\cdot)$ *such that*

$$\sup_{t\in\mathbb{R}+} \|u(t,\cdot) - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - \xi_j(t))\|_{\mathcal{H}} < B(\varepsilon + L^{-1/8}) \tag{20}$$

and

$$\sup_{t\in\mathbb{R}+} \|u(t,\cdot) - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - \xi_j(t))\|_{L^\infty} \lesssim \varepsilon^{2/3} + L^{-\frac{1}{12}} \ . \tag{21}$$

Moreover, for any $t \ge 0$ *and* $i \in [[1, N_+]]$ *(resp.* $i \in [[-N_-, -1]]$*),* $\xi_i(t)$ *is the only point of maximum (resp. minimum) of* $v(t)$ *on* $[\xi_i(t) - L/4, \xi_i(t) + L/4]$*.*

2. Global well-posedness results

We first recall some obvious estimates that will be useful in the sequel of this paper. Noticing that $p(x) = \frac{1}{2}e^{-|x|}$ satisfies $p * y = (1-\partial_x^2)^{-1} y$ for any $y \in H^{-1}(\mathbb{R})$ we easily get

$$\|u\|_{W^{1,1}} = \|p * (u - u_{xx})\|_{W^{1,1}} \lesssim \|u - u_{xx}\|_{\mathcal{M}}$$

and t

$$\|u_{xx}\|_{\mathcal{M}} \le \|u\|_{L^1} + \|u - u_{xx}\|_{\mathcal{M}}$$

which ensures that

$$Y \hookrightarrow \big\{ u \in W^{1,1}(\mathbb{R}) \text{ with } u_x \in \mathcal{B}V(\mathbb{R}) \big\} \ . \tag{22}$$

It is also worth noticing that for $u \in C_0^\infty(\mathbb{R})$, satisfying Hypothesis 1,

$$u(x) = \frac{1}{2}\int_{-\infty}^{x} e^{x'-x}(u - u_{xx})(x')dx' + \frac{1}{2}\int_{x}^{+\infty} e^{x-x'}(u - u_{xx})(x')dx' \tag{23}$$

and

$$u_x(x) = -\frac{1}{2}\int_{-\infty}^{x} e^{x'-x}(u - u_{xx})(x')dx' + \frac{1}{2}\int_{x}^{+\infty} e^{x-x'}(u - u_{xx})(x')dx' \ ,$$

so that for $x \le x_0$ we get

$$u_x(x) = u(x) - e^{-x}\int_{-\infty}^{x} e^{x'} y(x')\, dx' \ge u(x)$$

whereas for $x \ge x_0$ we get

$$u_x(x) = -u(x) + e^{x}\int_{x}^{+\infty} e^{-x'} y(x')\, dx' \ge -u(x)$$

Throughout this paper, we will denote $\{\rho_n\}_{n\ge 1}$ the mollifiers defined by

$$\rho_n = \Big(\int_{\mathbb{R}} \rho(\xi)\, d\xi\Big)^{-1} n\rho(n\cdot) \text{ with } \rho(x) = \begin{cases} e^{1/(x^2-1)} & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| \ge 1. \end{cases} \tag{24}$$

Following [19] we approximate $u \in Y$ satisfying Hypothesis 1 by the sequence of functions

$$u_n = p * y_n \text{ with } y_n = -(\rho_n * y^-)(\cdot + \frac{1}{n}) + (\rho_n * y^+)(\cdot - \frac{1}{n}) \text{ and } y = u - u_{xx}, \tag{25}$$

that belong to $Y \cap H^\infty(\mathbb{R})$ and satisfy Hypothesis 1 with the same x_0. It is not too hard to check that

$$\|y_n\|_{L^1} \le \|y\|_{\mathcal{M}}. \tag{26}$$

Moreover, noticing that

$$u_n = -\Big(\rho_n * (p * y^-)\Big)(\cdot + \frac{1}{n}) + \Big(\rho_n * (p * y^+)\Big)(\cdot - \frac{1}{n}),$$

with $p * y^\mp \in H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$, we infer that

$$u_n \to u \in H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})\ . \tag{27}$$

that ensures that for any $u \in Y$ satisfying Hypothesis 1 it holds

$$u_x \ge u \text{ on }]-\infty, x_0[\quad \text{and} \quad u_x \ge -u \text{ on }]x_0, +\infty[\ . \tag{28}$$

The following propositions briefly recall the global well-posedness results for the Cauchy problem of the DP equation (see for instance [9] and [15] for details of the proof) and its consequences.

Proposition 1. *(Strong solutions* [15], [9]*)*
Let $u_0 \in H^s(\mathbb{R})$ with $s \ge 3$. Then the initial value problem (4) *has a unique solution $u \in C\big([0,T]; H^s(\mathbb{R})\big) \cap C^1\big([0,T]; H^{s-1}(\mathbb{R})\big)$ where $T = T\big(\|u_0\|_{H^{\frac{3}{2}+}}\big) > 0$ and, for any $r > 0$, the map $u_0 \to u$ is continuous from $B(0,r)_{H^s}$ into $C\big([0, T(r); H^s(\mathbb{R})\big)$.*
Moreover, let $T^* > 0$ be the maximal time of existence of u in $H^s(\mathbb{R})$ then

$$T^* < +\infty \quad \Leftrightarrow \quad \liminf_{t \nearrow T^*} u_x = -\infty\ . \tag{29}$$

If furthermore $y_0 = u_0 - u_{0,xx} \in L^1(\mathbb{R})$ and u_0 satisfies Hypothesis 1 then $T^* = +\infty$ and $y = u - u_{xx} \in L^\infty_{loc}\big(\mathbb{R}_+; L^1(\mathbb{R})\big)$ with

$$\|y(t)\|_{L^1} \le e^{3t^2\|u_0\|_{L^2} + 2t\|u_0\|_{L^\infty}} \|y_0\|_{L^1}, \quad \forall t \in \mathbb{R}_+, \tag{30}$$

and

$$\int_{\mathbb{R}} y(t,x)\, dx = \int_{\mathbb{R}} y(0,x)\, dx, \quad \forall t \in \mathbb{R}_+\ . \tag{31}$$

Proposition 2. *(Global Weak Solution* [9]*)*
Let $u_0 \in Y$ satisfying Hypothesis 1 for some $x_0 \in \mathbb{R}$.

1. Uniqueness and global existence : (4) *has a unique solution*

$$u \in C\big(\mathbb{R}_+; H^1(\mathbb{R})\big) \cap C^1\big(\mathbb{R}_+; L^2(\mathbb{R})\big) \cap L^\infty_{loc}\big(\mathbb{R}_+; Y\big)\ .$$

$M(u) = \langle y, 1\rangle$, $E(u) = \langle y, v\rangle$ and $F(u)$ are conservation laws . Moreover, for any $t \in \mathbb{R}_+$, the density momentum $y(t)$ satisfies $\operatorname{supp} y^-(t) \subset]-\infty, x_0(t)]$ and $\operatorname{supp} y^+(t) \subset [x_0(t), +\infty[$ where $x_0(t) = q(t, x_0)$ with $q(\cdot,\cdot)$ defined by

$$\left\{ \begin{array}{rcll} q_t(t,x) & = & u(t, q(t,x)) & ,\ (t,x) \in \mathbb{R}^2 \\ q(0,x) & = & x & ,\ x \in \mathbb{R} \end{array} \right. . \tag{32}$$

2. Continuity with respect to initial data *: For any sequence $\{u_{0,n}\}$ bounded in Y that satisfy Hypothesis 1 and such that $u_{0,n} \to u_0$ in $H^1(\mathbb{R})$, the emanating sequence of solutions $\{u_n\}$ satisfies for any $T > 0$*

$$u_n \to u \ \text{in}\ C\big([0,T]; H^1(\mathbb{R})\big)\ . \tag{33}$$

and

$$(1 - \partial_x^2) u_n \underset{n\to\infty}{\rightharpoonup *} (1 - \partial_x^2) u \ \text{in}\ L^\infty\big(]0, T[; \mathcal{M}(\mathbb{R})\big) \quad . \tag{34}$$

Proof. Assertion 1. is proved in [9] except the conservation of $F(u)$. But this is clearly a direct consequence of the conservation of F for smooth solutions together with (33). So let us prove Assertion 2. We first assume that $\{u_{0,n}\}$ is the sequence defined in (25). In view of the conservation of $\mathcal{H}$ and (30), the sequence $\{u_n\}$ of smooth solutions to the DP equation emanating from $\{u_{0,n}\}$ is bounded in $C([0,T];H^1)\cap L^\infty(]0,T[;Y)$ for any fixed $T>0$. Therefore, there exists $w\in L^\infty(\mathbb{R}_+;H^1(\mathbb{R}))$ with $(1-\partial_x^2)w\in L^\infty_{loc}(\mathbb{R}_+;\mathcal{M}(\mathbb{R}))$ such that, for any $T>0$,

$$u_n \underset{n\to\infty}{\rightharpoonup} w\in L^\infty(]0,T[;H^1(\mathbb{R})) \quad\text{and}\quad (1-\partial_x^2)u_n \underset{n\to\infty}{\rightharpoonup}\ast\ (1-\partial_x^2)w \text{ in } L^\infty(]0,T[;\mathcal{M}(\mathbb{R}))\ .$$

Moreover, in view of (4), $\{\partial_t u_n\}$ is bounded in $L^\infty(]0,T[;L^2(\mathbb{R})\cap L^1(\mathbb{R}))$ and Helly's, Aubin-Lions compactness and Arzela-Ascoli theorems ensure that w is a solution to (4) that belongs to $C_w([0,T];H^1(\mathbb{R}))$ with $w(0)=u_0$. In particular, $w_t\in L^\infty(]0,T[;L^2(\mathbb{R}))$ and thus $w\in C([0,T];L^2(\mathbb{R}))$. Since $w\in L^\infty(]0,T[;H^{\frac{3}{2}-}(\mathbb{R}))$, this actually implies that $w\in C([0,T];H^{\frac{3}{2}-}(\mathbb{R}))$ and thus $w_t\in C([0,T];L^2(\mathbb{R}))$. Therefore, w belongs to the uniqueness class which ensures that $w=u$ and that (34) holds for this sequence. In particular passing to the limit in (30) we infer that for any $u_0\in Y$ satisfying Hypothesis 1 it holds

$$\|y(t)\|_{\mathcal{M}}\le e^{3t^2\|u_0\|_{L^2}+2t\|u_0\|_{L^\infty}}\|y_0\|_{\mathcal{M}},\forall t\in\mathbb{R}_+\ . \tag{35}$$

With (35) in hands, we can now proceed exactly in the same way but for any sequence $\{u_{0,n}\}$ bounded in Y that converges to u_0 in $H^1(\mathbb{R})$. This shows that (34) holds. Finally, the conservation of $E(\cdot)$ together with the weak convergence result in $C_w([0,T];H^1(\mathbb{R}))$ lead to a strong convergence result in $C([0,T];L^2(\mathbb{R}))$ that leads to (33) by using that $u\in L^\infty_{loc}(\mathbb{R}_+;H^{\frac{3}{2}-}(\mathbb{R}))$. ☐

In the sequel, we will make a constant use of the following properties of the flow-map $q(\cdot,\cdot)$ established for instance in [20] : Under the hypotheses of Proposition 2,

(1) The mapping $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$q_x(t,x)=exp\Big(\int_0^t u_x\big(s,q(s,x)\big)ds\Big)>0,\qquad \forall (t,x)\in\mathbb{R}_+\times\mathbb{R}. \tag{36}$$

(2) If moreover $u_0\in H^3(\mathbb{R})$ then

$$y\big(t,q(t,x)\big)q_x^3(t,x)=y_0(x),\qquad \forall (t,x)\in\mathbb{R}_+\times\mathbb{R}. \tag{37}$$

In particular, for all $t\ge 0$,

$$y\big(t,x_0(t))\big)=y\big(t,q(t,x_0)\big)=0,\qquad \forall (t,x)\in\mathbb{R}_+\times\mathbb{R}. \tag{38}$$

3. Some uniform L^∞-estimates

In [15] it is proven that as far as the solution to the DP equation stays smooth, its L^∞-norm can be bounded by a polynomial function of time with coefficients that depend only on the L^2 and L^∞-norm of the initial data. In this section we first improve this result under Hypothesis 1 by showing that the solution is then bounded in positive times by a constant that only depends on the L^2-norm of the initial data. This result is not directly needed in our work but we think that it has its own interest. In a second time we use the same type of arguments to prove that any function that is L^2-close to a peakon profile and satisfies Hypothesis 1, is actually L^∞-close to the peakon profile. This last result will be very useful for our work and will for instance enable us to prove that as far as u stays L^2-close to a translation of a peakon profile, the growth of the total variation of its momentum density can be control by an exponential function of the time but with a small constant in front of the time. This will be the aim of the last lemma of this section.

Lemma 1. *For any $u_0\in Y$ satisfying Hypothesis 1, the associated solution $u\in C(\mathbb{R}_+;H^1)$ to (4) given by Proposition 2 satisfies*

$$\|u(t)\|_{L^\infty(\mathbb{R})}\le 2(1+\sqrt{2})\|u_0\|_{\mathcal{H}},\quad \forall t\in\mathbb{R}_+. \tag{39}$$

Proof. We fix $t \in \mathbb{R}_+$, $x \in \mathbb{R}$ and denote by $\mathcal{E}(x)$ the integer part of x. Since $u(t,\cdot) \in H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$, the Mean-Value theorem for integrals together with (10) and the conservation of $\|\cdot\|_{\mathcal{H}}$ ensure that there exists $\eta \in \big[\mathcal{E}(x)-1, \mathcal{E}(x)\big]$ such that

$$u^2(t,\eta) = \int_{\mathcal{E}(x)-1}^{\mathcal{E}(x)} u^2(t,\theta)d\theta \le \|u(t,\cdot)\|^2_{L^2(\mathbb{R})} \le 4\|u(t,\cdot)\|^2_{\mathcal{H}} = 4\|u_0\|^2_{\mathcal{H}}.$$

Therefore, by (28) and (10), since $0 \le x-\eta \le 2$, one may write

$$\begin{aligned} u(t,x) &= u(t,\eta) + \int_\eta^x u_\theta(t,\theta)d\theta \ge -2\|u_0\|_{\mathcal{H}} - \int_\eta^x |u(t,\theta)|\, d\theta \ge -2\|u_0\|_{\mathcal{H}} - 2\sqrt{x-\eta}\|u_0\|_{\mathcal{H}} \\ &\ge -2(1+\sqrt{2})\|u_0\|_{\mathcal{H}}\ . \end{aligned} \tag{40}$$

Now, suppose that there exists $x_* \in \mathbb{R}$ such that $u(t,x_*) > 2(1+\sqrt{2})\|u_0\|_{\mathcal{H}}$. Then, on one side the Mean-Value theorem for integrals similarly ensures that there exists $\gamma \in \big[\mathcal{E}(x_*)+1, \mathcal{E}(x_*)+2\big]$ such that

$$u^2(t,\gamma) = \int_{\mathcal{E}(x_*)+1}^{\mathcal{E}(x_*)+2} u^2(t,\theta)d\theta \le \|u(t,\cdot)\|^2_{L^2(\mathbb{R})} \le 4\|u_0\|^2_{\mathcal{H}}\ .$$

On the other side, (28) again leads to

$$u(t,\gamma) = u(t,x_*) + \int_{x_*}^{\gamma} u_\theta(t,\theta)d\theta > 2(1+\sqrt{2})\|u_0\|_{\mathcal{H}} - 2\sqrt{\gamma - x_*}\|u_0\|_{\mathcal{H}} > 2\|u_0\|_{\mathcal{H}}\ . \tag{41}$$

The fact that the two above estimates are not compatible completes the proof of the lemma. $\square$

Lemma 2 (L^∞ approximations)**.** *Let $\psi \in W^{1,\infty}(\mathbb{R}) \cap L^2(\mathbb{R})$ and $u \in Y$, satisfying Hypothesis 1, then*

$$\|u-\psi\|_{L^\infty(\mathbb{R})} \le 2\|u-\psi\|^{2/3}_{\mathcal{H}}\big(1+\sqrt{2}\|u-\psi\|^{2/3}_{\mathcal{H}} + \|\psi\|_{L^\infty} + \|\psi'\|_{L^\infty}\big)\ . \tag{42}$$

In particular, for any $(c,r) \in \mathbb{R}^2$ it holds

$$\|u-\varphi_c(\cdot - r)\|_{L^\infty(\mathbb{R})} \le 2\|u-\varphi_c(\cdot-r)\|^{2/3}_{\mathcal{H}}\big(1+\sqrt{2}\|u-\varphi_c(\cdot-r)\|^{2/3}_{\mathcal{H}} + 2c\big)\ . \tag{43}$$

Proof. We first notice that (43) follows directly from (42) by taking $\psi = \varphi_c(\cdot - r)$ and using that $\|\varphi_c\|_{L^\infty} = \|\varphi_c'\|_{L^\infty} = c$.

Let us now prove (43). We set $\alpha = \|u-\psi\|^{2/3}_{\mathcal{H}}$. Fixing $x \in \mathbb{R}$, there exists $k \in \mathbb{Z}$ such $x \in \big[k\alpha, (k+1)\alpha\big[$. Therefore, applying the Mean-Value theorem on the interval $\big[(k-1)\alpha, k\alpha\big]$, we obtain that there exists $\eta \in \big[(k-1)\alpha, k\alpha\big]$ such that

$$\big[u(\eta)-\psi(\eta)\big]^2 = \frac{1}{\alpha}\int_{(k-1)\alpha}^{k\alpha} \big[u(\theta)-\psi(\theta)\big]^2 d\theta \le \frac{4}{\alpha}\|u-\psi\|^2_{\mathcal{H}} = 4\alpha^2. \tag{44}$$

Now, in view of (28), we get

$$u(x)-\psi(x) = u(\eta)-\psi(\eta) + \int_\eta^x [u_x(\theta)-\psi'(\theta)]d\theta \ge -2\alpha - \sqrt{2\alpha}\,\big\||u|+|\psi'|\big\|_{L^2(](k-1)\alpha,(k+1)\alpha[)}. \tag{45}$$

and the triangular inequality together with (10) yield

$$\big\||u|+|\psi'|\big\|_{L^2(](k-1)\alpha,(k+1)\alpha[)} \le \big\||u-\psi|+|\psi|+|\psi'|\big\|_{L^2(](k-1)\alpha,(k+1)\alpha[)} \le 2\|u-\psi\|_{\mathcal{H}} + \sqrt{2\alpha}\,(\|\psi\|_{L^\infty}+\|\psi'\|_{L^\infty}).$$

We thus eventually get

$$u(x)-\psi(x) \ge -2\alpha\big(1+\sqrt{2}\alpha + \|\psi\|_{L^\infty} + \|\psi'\|_{L^\infty}\big). \tag{46}$$

Now, suppose that there exists $x_* \in \mathbb{R}$ such that

$$u(x_*)-\psi(x_*) > 2\alpha\big(1+\sqrt{2}\alpha + \|\psi\|_{L^\infty} + \|\psi'\|_{L^\infty}\big)$$

Similarly, there exists $k_* \in \mathbb{R}$ such that $x_* \in \big[k_*\alpha, (k_*+1)\alpha\big[$ and applying the Mean-Value theorem for integrals on $\big[(k_*+1)\alpha, (k_*+2)\alpha\big]$ we obtain that there exists $\gamma \in \big[(k_*+1)\alpha, (k_*+2)\alpha\big]$ such that, on one hand,

$$\big[u(\gamma) - \psi(\gamma)\big]^2 = \frac{1}{\alpha}\int_{(k_*+1)\alpha}^{(k_*+2)\alpha} \big[u(\theta) - \psi(\theta)\big]^2 d\theta \leq 4\alpha^2 \ .$$

On the other hand, proceeding as above we get

$$\begin{aligned} u(\gamma) - \psi(\gamma) &= u(x_*) - \psi(x_*) + \int_{x_*}^{\gamma} [u_x(\theta) - \psi'(\theta)]d\theta \\ &> 2\alpha\Big(1 + \sqrt{2}\alpha + \|\psi\|_{L^\infty} + \|\psi'\|_{L^\infty}\Big) - \sqrt{2\alpha}\Big(2\alpha^{3/2} + \sqrt{2\alpha}(\|\psi\|_{L^\infty} + \|\psi'\|_{L^\infty})\Big) > 2\alpha \ . \end{aligned}$$

The incompatibility of the two above estimates completes the proof of the lemma. □

Lemma 3. *Let $u_0 \in Y$ satisfying Hypothesis 1 and $u \in C(\mathbb{R}_+; H^1) \cap L^\infty(\mathbb{R}_+; Y)$ be the associated solution to DP given by Proposition 2. If for some $c \geq 0$, $0 < \alpha < 1$ and $T > 0$ it holds*

$$\sup_{t\in[0,T]} \inf_{r\in\mathbb{R}} \|u(t,\cdot) - \varphi_c(\cdot - r)\|_{\mathcal{H}} \leq \alpha, \tag{47}$$

then

$$u(t) \geq -4\alpha^{\frac{2}{3}}(2+c), \quad \forall t \in [0,T] \tag{48}$$

and

$$\sup_{t\in[0,T]} \|y(t)\|_{\mathcal{M}} \leq \big(1 + 2e^{8\alpha^{\frac{2}{3}}(2+c)t}\big)\|y_0\|_{\mathcal{M}}. \tag{49}$$

Proof. According to Proposition 2, approximating u_0 by the sequence $u_{0,n}$ given by (25), it suffices to prove the result for smooth initial data $u_0 \in Y \cap H^\infty(\mathbb{R})$ satisfying Hypothesis 1. We notice that since $\varphi_c > 0$ on $\mathbb{R}$, (47) together with Lemma 2 ensure that for all $t \in [0,T]$,

$$u(t,\cdot) \geq -2\alpha^{2/3}\big(1 + \sqrt{2}\alpha^{2/3} + 2c\big) \geq -4\alpha^{\frac{2}{3}}(2+c) \quad \text{on } \mathbb{R} \ .$$

Therefore, according to (7), (32), (38) and (28), we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} y^-(t,x)dx = -\frac{d}{dt}\int_{-\infty}^{q(t,x_0)} y(t,x)dx &= 2\int_{-\infty}^{q(t,x_0)} u_x(t,x)y(t,x)dx \\ &\leq -2\int_{-\infty}^{q(t,x_0)} (-u(t,x))\,y(t,x)dx \\ &\leq 8\alpha^{\frac{2}{3}}(2+c)\int_{\mathbb{R}} y^-(t,x)dx. \end{aligned}$$

Hence, Grönwall's inequality yields

$$\int_{\mathbb{R}} y^-(t,x)dx \leq e^{8\alpha^{\frac{2}{3}}(2+c)t}\int_{\mathbb{R}} y_0^-(x)dx. \tag{50}$$

Moreover, since, according to Proposition 1, $M(u) = \displaystyle\int_{\mathbb{R}} y$ is conserved for positive times, it holds $\displaystyle\int_{\mathbb{R}} y^+(t,x)dx = \int_{\mathbb{R}} y_0(x)dx + \int_{\mathbb{R}} y^-(t,x)dx$ and thus

$$\|y(t,\cdot)\|_{L^1(\mathbb{R})} \leq \big(1 + 2e^{8\alpha^{\frac{2}{3}}(2+c)t}\big)\|y_0\|_{L^1(\mathbb{R})}. \tag{51}$$

□

4. A dynamic estimate on y^-

In this section we assume that $\sup_{t\in[0,T]}\inf_{r\in\mathbb{R}}\|u(t)-\varphi_c(\cdot-r)\|_{\mathcal{H}}<\varepsilon$ for some $T>0$ and some $0<\varepsilon<1$ small enough. Then we can construct a C^1-function $x:[0,T]\to\mathbb{R}$ such that $\sup_{t\in[0,T]}\|u(t)-\varphi_c(\cdot-x(t))\|_{\mathcal{H}}\lesssim\varepsilon$ and we study the behavior of y^- in an growing with time interval at the left of $x(t)$.

Lemma 4. *There exist $\tilde{\varepsilon}_0>0$, $0<\kappa_0<1$ and $K\geq 1$ such that if a solution $u\in C([0,T];H^1(\mathbb{R}))$ to (4) satisfies for some $c>0$ and some function $r:[0,T]\to\mathbb{R}$,*

$$\sup_{t\in[0,T]}\|u(t,\cdot)-\varphi_c(\cdot-r(t))\|_{\mathcal{H}}<c\tilde{\varepsilon}_0, \tag{52}$$

then there exist a unique function $x\colon[0,T]\longrightarrow\mathbb{R}$ such that

$$\sup_{t\in[0,T]}\big|x(t)-r(t)\big|\leq\kappa_0<\ln(3/2) \tag{53}$$

and

$$\int_{\mathbb{R}} v(t,x)\rho'(x-x(t))\,dx=0,\quad \forall t\in[0,T]\,. \tag{54}$$

where $v=(4-\partial_x^2)^{-1}u$ and $\rho=(4-\partial_x^2)^{-1}\varphi$. Moreover, $x(\cdot)\in C^1([0,T])$ with

$$\sup_{t\in[0,T]}\big|\dot{x}(t)-c\big|\leq\frac{c}{8}, \tag{55}$$

and if

$$\sup_{t\in[0,T]}\big\|u(t,\cdot)-\varphi_c(\cdot-r(t))\big\|_{\mathcal{H}}<\varepsilon, \tag{56}$$

for some $0<\varepsilon\leq c\tilde{\varepsilon}_0$ then

$$\sup_{t\in[0,T]}\big\|u(t,\cdot)-\varphi_c(\cdot-x(t))\big\|_{\mathcal{H}}\leq K\varepsilon\,. \tag{57}$$

Proof. We follow the same approach as in [10], by requiring an orthogonality condition on $v=(4-\partial_x^2)^{-1}u$ instead of u. This will be useful to get the C^1-regularity of $x(\cdot)$. In the sequel of the proof, we endow $H^2(\mathbb{R})$ with the norm (that is equivalent to the usual norm)

$$\|v\|_{H^2}^2:=\int_{\mathbb{R}}4v^2+5v_x^2+v_{xx}^2=\|(4-\partial_x^2)v\|_{\mathcal{H}}^2$$

where the last identity follows from (6). Let $0<\varepsilon<1$. For $r\in\mathbb{R}$ we introduce the function $Y\colon(-\varepsilon,\varepsilon)\times B_{H^2}\big(\rho(\cdot-r),\varepsilon\big)\longrightarrow\mathbb{R}$ defined by

$$Y(y,v)=\int_{\mathbb{R}}\big[v(t,x)-\rho(x-r-y)\big]\rho'(x-r-y)dx.$$

It is clearly that $Y\big(0,\rho(\cdot-r)\big)=0$ and that Y is of class C^1. Moreover, by integration by parts, it holds

$$\frac{\partial Y}{\partial y}(y,v)=-\int_{\mathbb{R}}v(t,x)\rho''(x-r-y)dx.$$

Hence, by integration by parts we may write

$$\frac{\partial Y}{\partial y}\big(0,\rho(\cdot-r)\big)=-\int_{\mathbb{R}}\rho(x-r)\partial_x^2\rho(x-r)dx=\|\partial_x\rho(\cdot-r)\|_{L^2(\mathbb{R})}^2=\frac{5}{54}\neq 0. \tag{58}$$

From the Implicit Function Theorem we deduce that there exist $\tilde{\varepsilon}_0>0$, $0<\kappa_0<\ln(3/2)$ and a C^1-function $y_r\colon B_{H^2}\big(\rho(\cdot-r),\tilde{\varepsilon}_0\big)\longrightarrow]-\kappa_0,\kappa_0[$ which is uniquely determined such that

$$Y\big(y_r(v),v\big)=Y\big(0,\rho(\cdot-r)\big)=0,\qquad\forall v\in B_{H^2}\big(\rho(\cdot-r),\tilde{\varepsilon}_0\big).$$

In particular, there exists $C_0>0$ such that if $v\in B_{H^2}\big(\rho(\cdot-r),\beta\big)$, with $0<\beta\leq\tilde{\varepsilon}_0$, then

$$\big|y_r(v(t,\cdot))\big|\leq C_0\beta. \tag{59}$$

Note that by a translation symmetry argument $\tilde{\varepsilon}_0$, κ_0, and C_0 are independent of $r \in \mathbb{R}$. Therefore, by uniqueness, we can define a C^1-mapping $\tilde{x}\colon \bigcup_{r\in\mathbb{R}} B_{H^2}\big(\rho(\cdot - r), \tilde{\varepsilon}_0\big) \longrightarrow]-\kappa_0, \kappa_0[$ by setting

$$\tilde{x}(v) = r + y_r(v) \qquad \forall v \in B_{H^2}\big(\rho(\cdot - r), \tilde{\varepsilon}_0\big).$$

Now, according to (52), it holds $\{\frac{1}{c}v(t,), [0,T]\} \subset \cup_{z\in\mathbb{R}} B_{H^2}(\rho(\cdot - z), \tilde{\varepsilon}_0)$ so that we can define the function $x(\cdot)$ on $\mathbb{R}$ by setting $x(t) = \tilde{x}(v(t))$. By construction $x(\cdot)$ satisfies (53)-(54). Moreover, (56) together with (59) ensure that for any $c > 0$ and any $0 < \varepsilon < c\tilde{\varepsilon}_0$, it holds

$$\|\frac{1}{c}u(t) - \varphi(\cdot - x(t))\|_{\mathcal{H}} \le (\frac{\varepsilon}{c}) + \sup_{|z|\le C_0\frac{\varepsilon}{c}} \|\varphi - \varphi(\cdot - z))\|_{\mathcal{H}} \lesssim \frac{\varepsilon}{c} \tag{60}$$

which proves (57).

In view of (4), any solution $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ of (D-P) satisfies $u_t \in C(\mathbb{R}; L^2(\mathbb{R}))$ and thus belongs to $C^1(\mathbb{R}; L^2(\mathbb{R}))$. This ensures that $v \in C^1(\mathbb{R}; H^2(\mathbb{R}))$ so that the mapping $t \mapsto x(t) = \tilde{x}(v(t))$ is of class C^1 on $\mathbb{R}$.

Now, we notice that applying the operator $(4 - \partial_x^2)^{-1}$ to the two members of (4) and using that

$$(4 - \partial_x^2)^{-1}(1 - \partial_x^2)^{-1} = \frac{1}{3}(1 - \partial_x^2)^{-1} - \frac{1}{3}(4 - \partial_x^2)^{-1} \ , \tag{61}$$

we get that v satisfies

$$v_t = -\frac{1}{2}\partial_x(1 - \partial_x^2)^{-1}u^2 \ . \tag{62}$$

On the other hand, setting $R(t,\cdot) = c\rho(\cdot - x(t))$ and $w = v - R$ and differentiating (54) with respect to time we get

$$\begin{aligned}
\int_{\mathbb{R}} w_t \rho'(\cdot - x(t)) &= \dot{x}(t) \int_{\mathbb{R}} w\, \rho''(\cdot - x(t)) \\
&= -\dot{x}(t) \int_{\mathbb{R}} \partial_x w\, \rho'(\cdot - x(t)) \\
&= (\dot{x}(t) - c) O(\|w\|_{H^1}) + c\, O(\|w\|_{H^1}) \ .
\end{aligned} \tag{63}$$

Substituting v by $w + R$ in (62) and using that R satisfies

$$\partial_t R + (\dot{x} - c)\partial_x R = -\frac{1}{2}\partial_x(1 - \partial_x^2)^{-1}\varphi_c^2(\cdot - x(t)) \ ,$$

we infer that w satisfies

$$w_t - (\dot{x} - c)\partial_x R = -\frac{1}{2}\partial_x(1 - \partial_x^2)^{-1}\Big(u^2 - \varphi_c^2\Big) = -\frac{1}{2}\partial_x(1 - \partial_x^2)^{-1}\Big((u - \varphi_c)(u + \varphi_c)\Big) \ .$$

Taking the L^2-scalar product of this last equality with $\rho'(\cdot - x(t))$ and using (63) together with (52) and (57) we get, for all $t \in [0,T]$,

$$\Big|(\dot{x}(t) - c)\Big(\int_{\mathbb{R}} \partial_x R(t,\cdot)\, \rho'(\cdot - x(t)) + c\, O(\|w\|_{H^1})\Big)\Big| \le O(\|w\|_{H^1}) + O(\|u - \varphi_c(\cdot - x(t)\|_{\mathcal{H}}) \lesssim Kc\, \tilde{\varepsilon}_0 \ .$$

Therefore, by recalling (58) and possibly decreasing the value of $\tilde{\varepsilon}_0 > 0$ so that $K\tilde{\varepsilon}_0 \ll 1$, we obtain (55). □

Proposition 3. *There exists $\varepsilon_0 > 0$ such that for any $u_0 \in Y \cap H^\infty(\mathbb{R})$ satisfying Hypothesis 1, if the solution $u \in C(\mathbb{R}_+; H^\infty(\mathbb{R})\big)$ emanating from u_0 satisfies for some $c > 0$, $T > 0$ and some function $r : [0,T] \to \mathbb{R}$,*

$$\sup_{t\in[0,T]} \|u(t,\cdot) - \varphi_c(\cdot - r(t))\|_{\mathcal{H}} < \varepsilon_0(1 \wedge c^2), \tag{64}$$

then for all $t \in [0,T]$,

$$\|y^-(t,\cdot)\|_{L^1(]r(t) - \frac{1}{16}ct, +\infty[)} \le e^{-ct/8}\|y_0\|_{L^1(\mathbb{R})}. \tag{65}$$

where $y^- = \max(-y, 0)$ and $x(\cdot)$ is the C^1-function constructed in Lemma 4.

Proof. Let $\tilde{\varepsilon}_0 > 0$ and $K \geq 1$ be the universal constants that appears in the statement of Lemma 4. Assuming (56) with

$$\varepsilon < \min(c\tilde{\varepsilon}_0, 10^{-20}(1 \wedge c^2)/K) \leq \frac{10^{-20} \wedge \tilde{\varepsilon}_0}{K}(1 \wedge c^2) \ ,$$

(57), Lemma 2 ensure that

$$\sup_{t\in[0,T]} \|u(t,\cdot) - \varphi_c(\cdot - x(t))\|_{L^\infty} \leq 10^{-5} c \ . \tag{66}$$

where $x(\cdot)$ is the C^1-function constructed in Lemma 4. Therefore, setting

$$\varepsilon_0 := \frac{10^{-20} \wedge \tilde{\varepsilon}_0}{K} \ , \tag{67}$$

(64) ensures that (66) holds.

Let $t \in [0,T]$, we separate two possible cases according to the distance between $x_0(t/2)$ and $x(t/2)$, where $x_0(\cdot)$ is defined in Proposition 2.

Case 1:

$$x_0(t/2) < x(t/2) - \ln(3/2). \tag{68}$$

In view of (66) and the monotony of φ on $\mathbb{R}_-$, it holds

$$u(\tau,x) \leq \varphi_c\big(-\ln(3/2)\big) + \frac{c}{16} = \frac{2}{3}c + \frac{1}{16}c \leq \frac{3}{4}c, \quad \forall x \leq x(\tau) - \ln(3/2) \text{ with } \tau \in [0,T] \ . \tag{69}$$

In particular (68) and (32) lead to

$$\dot{x}_0(t/2) = u\big(t/2, x_0(t/2)\big) \leq \frac{3}{4}c. \tag{70}$$

Therefore, since (55) forces $\dot{x}(t) \geq 7c/8$ on $[0,T]$, a classical continuity argument ensures that $x_0(\cdot) < x(\cdot) - \ln(3/2)$ on $[t/2,T]$ and thus $\dot{x}_0(\cdot) \leq \frac{3}{4}c$ on $[t/2,T]$. It follows from (53) that

$$r(t) - x_0(t) \geq x(t) - x_0(t) - \ln(3/2) = \int_{t/2}^{t} \big[\dot{x}(\theta) - \dot{x}_0(\theta)\big] d\theta + x(t/2) - x_0(t/2) - \ln(3/2) \geq \frac{c}{16} t \ .$$

This proves that $y^-(t,\cdot) = 0$ on $]r(t) - \frac{1}{16}ct, +\infty[$ and thus that (65) holds in this case.

Case 2:

$$x_0(t/2) \geq x(t/2) - \ln(3/2). \tag{71}$$

Then we first claim that

$$x_0(\tau) \geq x(\tau) - \ln(3/2) \qquad \forall \tau \in [0, t/2]. \tag{72}$$

Indeed, assuming the contrary, we would get as above that $x_0(\cdot) < x(\cdot) - \ln(3/2)$ on $[\tau, T]$ that would contradicts (71). Second, we notice that (66) ensures that

$$u\big(\tau, x(\tau) - \ln(3)\big) \geq \varphi_c(-\ln(3)) - \frac{c}{16} \geq \frac{c}{4}, \quad \forall \tau \in [0,T] \ . \tag{73}$$

Since (28) forces $u_x(\tau) \geq u(\tau)$ on $]-\infty, x_0(\tau)]$ for any $\tau \in \mathbb{R}_+$, (72)-(73) then ensure that $u(\tau)$ is increasing on $[x_0(\tau) - \ln(2), x_0(\tau)]$ and

$$u_x(\tau,x) \geq u(\tau,x) \geq \frac{c}{4}, \quad \forall (\tau,x) \in [0,T] \times [x_0(\tau) - \ln 2, x_0(\tau)] \ . \tag{74}$$

Now, in this case we divide the proof into two steps.

Step: 1. The aim of this step is to prove the following estimate on $y(t/2)$:

$$\left| \int_{x_0(t/2)-\ln 2}^{x_0(t/2)} y(t/2,s) ds \right| \leq e^{-\frac{1}{4}ct} \|y_0\|_{L^1(\mathbb{R})}. \tag{75}$$

For $\tau \in \mathbb{R}_+$, we denote by $q^{-1}(\tau,\cdot)$ the inverse mapping of $q(\tau,\cdot)$. Then, the change of variables along the flow $\theta = q^{-1}(t/2,s)$ leads to

$$\int_{x_0(t/2)-\ln 2}^{x_0(t/2)} y(t/2,s) ds = \int_{q^{-1}(t/2,x_0(t/2)-\ln 2)}^{q^{-1}(t/2,x_0(t/2))} y\big(t/2, q(t/2,\theta)\big) q_x(t/2,\theta) \, d\theta. \tag{76}$$

Since $x_0(\tau)=q(\tau,x_0)$ it clearly holds $x_0=q^{-1}(t/2,x_0(t/2))$ and (74) together with (32) force

$$\partial_t q\big(\tau,q^{-1}(t/2,x)\big)\le \dot x_0(\tau),\quad \forall\,(\tau,x)\in[0,t/2]\times[x_0(t/2)-\ln 2,x_0(t/2)]\ .$$

This ensures that for all $\tau\in[0,t/2]$,

$$0<x_0(\tau)-q(\tau,q^{-1}(t/2,x_0(t/2)-\ln 2))\le x_0(t/2)-q(t/2,q^{-1}(t/2,x_0(t/2)-\ln 2))=\ln 2 \tag{77}$$

In particular, for any $\theta\in[q^{-1}(t/2,x_0(t/2)-\ln 2),x_0]$ and any $\tau\in[0,t/2]$, it holds $q(\tau,\theta)\in[x_0(\tau)-\ln 2,x_0(\tau)]$ and (74) yields

$$u_x(\tau,q(\tau,\theta))\ge u(\tau,q(\tau,\theta))\ge c/4\ .$$

In view of (36) we thus deduce that

$$q_x(t/2,\theta)=\exp\Big(\int_0^{t/2}u_x(\tau,q(\tau,\theta))\,d\tau\Big)\ge \exp(\frac{c}{8}t)\ .$$

Plugging this estimate in (76), using (37), (77) and that $y(\tau,\cdot)\le 0$ on $]-\infty,x_0(\tau)]$ for $\tau\ge 0$, we eventually get

$$\begin{aligned}\int_{x_0(t/2)-\ln 2}^{x_0(t/2)}y(t/2,s)ds &\ge e^{-\frac{c}{4}t}\int_{q^{-1}(t/2,x_0(t/2)-\ln 2)}^{q^{-1}(t/2,x_0(t/2))=x_0}y\big(t/2,q(t/2,\theta)\big)q_x^3(t/2,\theta)\,d\theta\\ &\ge e^{-\frac{c}{4}t}\int_{x_0-\ln 2}^{x_0}y(0,\theta)\,d\theta\end{aligned}$$

which proves (75).

Step: 2. In this step, we prove that

$$\Big|\int_{x(t)-\ln(3/2)-\frac{c}{16}t}^{x_0(t)}y(t,s)\,ds\Big|\le e^{ct/8}\Big|\int_{x_0(t/2)-\ln 2}^{x_0(t/2)}y(t/2,s)\,ds\Big| \tag{78}$$

Clearly, (78) combined with (75) and (53) prove that (65) also holds in this case which completes the proof of the proposition.

First, for any $t_1\ge 0$ we define the function $q_{t_1}(\cdot,\cdot)$ on $\mathbb{R}_+\times\mathbb{R}$ as follows

$$\begin{cases}\partial_t q_{t_1}(t,x)=u(t,q_{t_1}(t,x)), & \forall(t,x)\in\mathbb{R}_+\times\mathbb{R},\\ q_{t_1}(t_1,x)=x, & x\in\mathbb{R}.\end{cases} \tag{79}$$

The mapping $q_{t_1}(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ and we denote by $q_{t_1}^{-1}(t,\cdot)$ it inverse mapping. As in (36) we have

$$\partial_x q_{t_1}(t,x)=exp\Big(\int_{t_1}^t u_x\big(s,q_{t_1}(s,x)\big)ds\Big)>0,\qquad \forall(t,x)\in\mathbb{R}_+\times\mathbb{R},$$

and

$$y\big(t,q_{t_1}(t,x)\big)(\partial_x q_{t_1})^3(t,x)=y(t_1,x),\qquad \forall(t,x)\in\mathbb{R}_+\times\mathbb{R}. \tag{80}$$

In particular, (48), (28) together with (66) ensure that for any $\tau\in[t/2,t]$ and any $x\le x_0(t/2)$,

$$\partial_x q_{t/2}(\tau,x)\ge exp\Big(-\int_{t/2}^t 2^{-5}c\,ds\Big)\ge e^{-2^{-4}ct}\quad . \tag{81}$$

Using the change of variables $\theta=q_{t/2}^{-1}(t,s)$ we eventually get

$$\int_{q_{t/2}(t,x_0(t/2)-\ln 2)}^{x_0(t)}y(t,s)\,ds=\int_{x_0(t/2)-\ln 2}^{x_0(t/2)}y(t,q_{t/2}(t,\theta))\partial_x q_{t/2}(t,\theta)\,d\theta$$

and (80)-(81) lead to

$$\int_{q_{t/2}(t,x_0(t/2)-\ln 2)}^{x_0(t)}y(t,s)\,ds\ge e^{ct/8}\int_{x_0(t/2)-\ln 2}^{x_0(t/2)}y(t/2,\theta)\,d\theta \tag{82}$$

Now, we notice that (66) forces

$$x_0(\tau)\le x(\tau)+\ln(4/3),\quad \forall\tau\in[0,T]\ . \tag{83}$$

Indeed, otherwise since $u(\tau,x(\tau)) \geq c - \frac{c}{16}$ and $u_x(\tau,\cdot) \geq u(\tau,\cdot)$ on $]-\infty, x_0(\tau)]$ this would imply that $u(\tau, x(\tau)+\ln(4/3)) \geq \frac{15}{16}c$ that is not compatible with $\varphi_c(\ln(4/3)) = \frac{4}{3}c$ and (64). From (83) we deduce that for all $\tau \in [0,T]$,

$$x_0(\tau) - \ln(2) \leq x(\tau) - \ln(3/2) \tag{84}$$

and thus

$$u(\tau, x_0(\tau) - \ln(2)) \leq \varphi_c(x_0(\tau) - x(\tau) - \ln(2)) + \frac{c}{16} \leq \varphi_c(-\ln(3/2)) + \frac{c}{16} \leq \frac{3c}{4} \ .$$

Combining this last inequality at $\tau = t/2$ with (79), (55) and a continuity argument we infer that

$$\dot{x}(\tau) - \partial_t q_{t/2}(\tau, x_0(t/2) - \ln(2)) \geq \frac{c}{8}, \quad \forall \tau \in [t/2, T] \ ,$$

which yields

$$q_{t/2}(t, x_0(t/2) - \ln(2)) \leq x(t) - \ln(3/2) - \frac{c}{16}t \ . \tag{85}$$

Combining (82) and (85), (78) follows. □

Corollary 1. *Under the same hypotheses as in Proposition 3, for all $t \in [0,T]$, it holds*

$$u(t,\cdot) - 6v(t,\cdot) \leq e^{9-\frac{ct}{32}} \|y_0\|_{L^1(\mathbb{R})} \quad on\ ]r(t)-8, +\infty[\ , \tag{86}$$

where $v = (4 - \partial_x)^{-1}u$.

Proof. By (61), it hods

$$\begin{aligned} 6v - u &= (1-\partial_x^2)^{-1}y - 2(4-\partial_x^2)^{-1}y = \frac{1}{2}e^{-|\cdot|} * y - \frac{1}{2}e^{-2|\cdot|} * y \\ &= \frac{1}{2}(e^{-|\cdot|} - e^{-2|\cdot|}) * y \\ &\geq -\frac{1}{2}(e^{-|\cdot|} - e^{-2|\cdot|}) * y^- \geq -\frac{1}{2}e^{-|\cdot|} * y^- . \end{aligned} \tag{87}$$

Therefore, for $x \geq r(t) - 8$, (49), (65) and (67) lead to

$$\begin{aligned} 6v(x) - u(x) &\geq -\frac{1}{2}\int_{-\infty}^{r(t)-\frac{c}{16}t} e^{-|x-z|}y^-(z)\,dz - \frac{1}{2}\int_{r(t)-\frac{c}{16}t}^{+\infty} e^{-|x-z|}y^-(z)\,dz \\ &\geq -\frac{1}{2}e^{0\wedge(8-\frac{c}{16}t)}(1+2e^{2^{-5}ct})\|y_0\|_{L^1} - \frac{1}{2}e^{-ct/8}\|y_0\|_{L^1(\mathbb{R})} \\ &\geq -e^{9-\frac{ct}{32}}\|y_0\|_{L^1(\mathbb{R})} \ . \end{aligned}$$

□

5. Proof of Theorem 1

Before starting the proof, we need the two following lemmas that will help us to rewrite the problem in a slightly different way. The next lemma ensures that the distance in $\mathcal{H}$ to the translations of φ_c is minimized for any point of maximum of $v = (4 - \partial_x^2)^{-1}u$.

Lemma 5 (Quadratic Identity [13])**.** *For any $u \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$, it holds*

$$E(u) - E(\varphi_c) = \|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}}^2 + 4c\left(v(\xi) - \frac{c}{6}\right), \tag{88}$$

where $v = (4-\partial_x^2)^{-1}u$ and ξ is any point where v reaches its maximum.

We will also need the following lemma that is implicitly contained in [10].

Lemma 6. *Let $u \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ such that*

$$\|u - \varphi_c(\cdot - r)\|_{L^\infty} \leq 10^{-5}\ c \tag{89}$$

for some $c > 0$ and some $r \in \mathbb{R}$. Then $v = (4-\partial_x^2)^{-1}u$ has got a unique point of maximum ξ on $\mathbb{R}$ and

$$\|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq \|u - \varphi_c(\cdot - r)\|_{\mathcal{H}} \ . \tag{90}$$

Finally, $\xi \in \Theta_r = [r-6.7, r+6.7]$, is the only critical point of v in Θ_r and

$$\sup_{x\notin\Theta_r} \big(|u(x)|, |v(x)|, |v_x(x)|\big) \le \frac{c}{100} \; . \tag{91}$$

Proof. Let us first recall that $v-\rho_c = \frac{1}{4}e^{-2|\cdot|} * (u-\varphi_c)$ so that Young convolution inequalities yield

$$\|v-\rho_c\|_{L^\infty} \le \|\frac{1}{4}e^{-2|\cdot|}\|_{L^1}\|u-\varphi_c\|_{L^\infty} \le \frac{1}{4}\|u-\varphi_c\|_{L^\infty} \quad \text{and} \quad \|(v-\rho_c)'\|_{L^\infty} \le \frac{1}{2}\|u-\varphi_c\|_{L^\infty} \; . \tag{92}$$

Moreover, $(v-\rho_c)'' = 4(v-\rho_c)-(u-\varphi_c)$ leads to

$$\|(v-\rho_c)''\|_{L^\infty} \le 2\|u-\varphi_c\|_{L^\infty} \; .$$

Now, the crucial observations in [11] are that

$$\rho'' \le \frac{\sqrt{2}-2}{6} \text{ on } \mathcal{V}_0, \quad \rho'(x) = -\rho'(-x) \ge 10^{-4}, \; \forall x \in \Theta_0/\mathcal{V}_0 \; , \tag{93}$$

where, $\forall r>0$, $\mathcal{V}_r = [r-\ln\sqrt{2}, r+\ln\sqrt{2}]$. Therefore, (89) together with (93) ensure that v' is strictly decreasing on $\mathcal{V}_r$ and that $v'>0$ on $[r-6.7, r-\ln\sqrt{2}]$ and $v'<0$ on $[r+\ln\sqrt{2}, r+6.7]$. This proves that v has got a unique critical point ξ on Θ_r that is a local maximum and that $\xi \in \mathcal{V}_r \subset \Theta_r$. Moreover $\rho(0) = 1/6$ together with the direct estimates

$$\rho \vee |\rho'| \le 5\times 10^{-4} \text{ on } \mathbb{R}/\Theta_0 \quad \text{and} \quad \varphi \vee |\varphi'| \le 5\times 10^{-3} \text{ on } \mathbb{R}/\Theta_0 \; , \tag{94}$$

ensure that this is actually the unique point of maximum of v on $\mathbb{R}$. This proves the first part of (91) whereas the second part follows again from (94). Finally, (90) follows directly from Lemma 5 together with the fact $v(\xi)$ is the maximum of v on $\mathbb{R}$. $\square$

Now, let us recall that, by (25), we can approximate any $u_0 \in Y$ satisfying Hypothesis 1 by a sequence $\{u_{0,n}\} \subset Y \cap H^\infty(\mathbb{R})$ satisfying Hypothesis 1 such that

$$u_{0,n} \to u_0 \text{ in } H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R}) \quad \text{and} \quad \|y_n\|_{L^1} \le \|y\|_{\mathcal{M}} \; , \; \forall n \in \mathbb{N}.$$

Therefore the continuity with respect to initial data in Proposition 2 ensures that to prove Theorem 1 we can reduce ourself to initial data $u_0 \in Y \cap H^\infty$.

Let ε_0 be the universal constant defined in (67) and let us fix

$$0 < \varepsilon < \varepsilon_0 \frac{1\wedge c^2}{(2+c)^3} \; . \tag{95}$$

Let us also fix $A>0$. From the continuity with respect to initial data (33) at φ_c, the fact that $t \mapsto \varphi_c(\cdot - ct)$ is an exact solution and the translation symmetry of the (D-P) equation, there exists

$$0 < \delta = \delta(A, \varepsilon, c) \le \varepsilon^4 \tag{96}$$

such that for any $u_0 \in Y$ satisfying Hypothesis 1 and (14)-(13) with A and δ, it holds

$$\|u(t) - \varphi_c(x-ct)\|_{\mathcal{H}} \le 2(2+c)\,\varepsilon, \quad \forall t \in [0, T_\varepsilon], \text{with} \quad T_\varepsilon = \max\Big(0, \frac{32}{c}\big(9+\ln(A/\varepsilon^2)\big)\Big) \; , \tag{97}$$

where $u \in C(\mathbb{R}_+; H^1(\mathbb{R}))$ is the solution of the (D-P) equation emanating from u_0. So let $u_0 \in Y \cap H^\infty(\mathbb{R})$ that satisfies Hypothesis 1 and (14)-(13) with A and δ. (97) together with the definition (67) of ε_0 and Lemma 2 then ensure that

$$\|u(t) - \varphi_c(x-ct)\|_{L^\infty} < 10^{-5}c, \quad \forall t \in [0, T_\varepsilon], \tag{98}$$

and Lemma 6 then ensures that

$$\|u(t) - \varphi_c(x-\xi(t))\|_{\mathcal{H}} \le 2(2+c)\,\varepsilon \; , \quad \forall t \in [0, T_\varepsilon], \tag{99}$$

where $\xi(t)$ is the only point where $v(t) = (4-\partial_x^2)^{-1}u(t)$ reaches its maximum.

By a continuity argument it remains to prove that for any $T \ge T_\varepsilon$, if

$$\inf_{r\in\mathbb{R}} \|u(t) - \varphi_c(x-r)\|_{\mathcal{H}} \le 3(2+c)\,\varepsilon \quad \text{on} \quad [0,T] \tag{100}$$

then $v(T) = (4-\partial_x^2)^{-1}u(T)$ reaches its maximum on $\mathbb{R}$ at a unique point $\xi(T)$ and

$$\|u(T) - \varphi_c(x-\xi(T))\|_{\mathcal{H}} \le 2(2+c)\,\varepsilon \; . \tag{101}$$

At this stage it is worth noticing that, as above, (100) together with the definition (67) of ε_0 and Lemma 2 ensure that

$$\inf_{r\in\mathbb{R}} \|u(t)-\varphi_c(x-r)\|_{L^\infty} \le 10^{-5}c, \quad \forall t\in[0,T] \ .$$

Therefore applying Lemma 6 and again Lemma 2 we obtain that

$$\|u(t)-\varphi_c(x-\xi(t))\|_{\mathcal{H}} \le 3(2+c)\,\varepsilon \quad \text{and} \quad \|u(t)-\varphi_c(x-\xi(t))\|_{L^\infty} \le 10^{-5}c, \quad \forall t\in[0,T], \tag{102}$$

where $\xi(t)$ is the only point where $v(t)=(4-\partial_x^2)^{-1}u(t)$ reaches its maximum. Moreover, (100) together with (95), (67), Corollary 1 and the definition of T_ε in (97) then ensure that

$$u(t,\cdot)-6v(t,\cdot)\le \varepsilon^2 \quad \text{on} \quad \Theta_{\xi(t)}, \quad \forall t\in[0,T] \ . \tag{103}$$

To prove (101), we follow closely the proof in [10], keeping (103) in hands. The idea comes back to [4] and consists in constructing two functions g and h that permits to link in a good way $E(u)$, $F(u)$ and the maximum M of $v=(4-\partial_x^2)^{-1}u$. This was first implement in [13] for the (DP)-equation under the additional hypothesis that the momentum density of the initial data is non negative.

Lemma 7 (See [13])**.** *Let $u\in L^2(\mathbb{R})$ and $v=(4-\partial_x^2)^{-1}u\in H^2(\mathbb{R})$. Denote by $M=\max_{\mathbb{R}} v(\cdot)=v(\xi)$ and define the function g by*

$$g(x)=\begin{cases} 2v(x)+v_{xx}(x)-3v_x(x)=u(x)-6v_x(x)+12v(x), & \forall x\le\xi, \\ 2v(x)+v_{xx}(x)+3v_x(x)=u(x)+6v_x(x)+12v(x), & \forall x\ge\xi. \end{cases} \tag{104}$$

Then it holds

$$\int_{\mathbb{R}} g^2(x)dx = E(u)-12M^2, \tag{105}$$

and

$$\int_{\mathbb{R}} g^2(x)dx = \left\|u-\varphi_c(\cdot-\xi)\right\|_{\mathcal{H}}^2 - 12\left(\frac{c}{6}-M\right)^2 \le \left\|u-\varphi_c(\cdot-\xi)\right\|_{\mathcal{H}}^2 \ . \tag{106}$$

Proof. The first identity is proven in [13] by combining integration by parts and the fact that $v_x(\xi)=0$. To prove the second identity we remark that by the definition of ρ_c in (9), it holds

$$\begin{cases} 2\rho_c(\cdot-\xi)-\rho_c''(\cdot-\xi)+3\rho_c'(\cdot-\xi)=0, & \forall x\le\xi, \\ 2\rho_c(\cdot-\xi)-\rho_c''(\cdot-\xi)-3\rho_c'(\cdot-\xi)=0, & \forall x\ge\xi. \end{cases}$$

Therefore, setting $w=v-\rho_c(\cdot-\xi)=(4-\partial_x^2)^{-1}[u-\varphi_c(\cdot-\xi)]$ one may rewrite g as

$$g=\begin{cases} 2w+w_{xx}-3w_x \text{ on }]-\infty,\xi], \\ 2w+w_{xx}+3w_x \text{ on } [\xi,+\infty[\end{cases} \tag{107}$$

and (106) follows by applying (105) with u replaced by $u-\varphi_c(\cdot-\xi)$. □

Lemma 8 (See [13])**.** *Let $u\in L^2(\mathbb{R})$ and $v=(4-\partial_x^2)^{-1}u\in H^2(\mathbb{R})$. Denote by $M=\max_{\mathbb{R}} v(\cdot)=v(\xi)$ and define the function h by*

$$h(x)=\begin{cases} -v_{xx}(x)-6v_x(x)+16v(x), & \forall x\le\xi. \\ -v_{xx}(x)+6v_x(x)+16v(x), & \forall x\ge\xi. \end{cases} \tag{108}$$

Then, it holds

$$F(u)-144M^3=\int_{\mathbb{R}} h(x)g^2(x)dx. \tag{109}$$

Gathering Lemmas 5, 7 and 8 and making use of (103) we derive the crucial relation that linked $E(u)$, $F(u)$ and the maximum M of $v=(4-\partial_x^2)^{-1}u$.

Lemma 9. *Let $\varepsilon > 0$ and $u \in L^2(\mathbb{R})$ be such that $v = (4-\partial_x^2)^{-1}u$ has got a unique point ξ of maximum on $\mathbb{R}$ with*

$$\|u-\varphi_c(\cdot-\xi)\|_{\mathcal{H}} \le 3(2+\varepsilon)\varepsilon, \quad \|u-\varphi_c(\cdot-\xi)\|_{L^\infty} \le 10^{-5}c \quad \text{and } u-6v \le \varepsilon^2 \quad \text{on} \quad \Theta_\xi \quad . \tag{110}$$

Then, setting $M = v(\xi)$, it holds

$$M^3 - \frac{1}{4}E(u)M + \frac{1}{72}F(u) \le \frac{(2+c)^2}{8}\varepsilon^4 \ . \tag{111}$$

Proof. The key is to show that the function h defined in Lemma 8 satisfies $h \le 18M + \varepsilon^2$ on $\mathbb{R}$. We notice that h may be rewritten as

$$h(x) = \begin{cases} u(x) - 6v_x(x) + 12v(x), & \forall x \le \xi. \\ u(x) + 6v_x(x) + 12v(x), & \forall x \ge \xi. \end{cases}$$

and that (92) together with the second inequality in (110) force

$$|M - c/6| \le 10^{-5}c \ . \tag{112}$$

Moreover, Lemma 6 ensures that $v_x > 0$ on $]\xi - 6.7, \xi[$ and $v_x < 0$ on $]\xi, \xi + 6.7[$.

We divide $\mathbb{R}$ into three intervals. For $x \in \mathbb{R}/\Theta_\xi$, (91) with $r = \xi$ and then (112) ensure that

$$h(x) \le |u(x)| + 6|v_x(x)| + 12|v(x)| \le \frac{19c}{100} \le 18M. \tag{113}$$

For $\xi - 6.7 < x < \xi$, then $v_x \ge 0$ and using that $u - 6v \le \varepsilon^2$ on Θ_ξ, we get

$$h(x) \le 18M + \varepsilon^2.$$

If $\xi < x < \xi + 6.7$, then $v_x \le 0$ and using that $u - 6v \le \varepsilon^2$ on Θ_ξ, we get

$$h(x) \le 18M + \varepsilon^2.$$

Therefore it holds,

$$h \le 18M + \varepsilon^2 \quad \text{on} \quad \mathbb{R} \ .$$

Combining (105), (106), (109) and the first inequality in (110), one eventually gets

$$\begin{aligned} F(u) - 144M^3 &= \int_{\mathbb{R}} h(x)g^2(x)dx \le 18M\big(E(u) - 12M^2\big) + \varepsilon^2\|u-\varphi_c(\cdot-\xi)\|_{\mathcal{H}}^2 \\ &\le 18ME(u) - 72M^3 + 9(2+c)^2\varepsilon^4 \ . \end{aligned}$$

that completes the proof of the lemma. □

Finally, we will need the following lemma that links the distance between $F(u_0)$ and $F(\varphi_c)$ to the distance between u_0 and φ_c in $L^2(\mathbb{R})$.

Lemma 10. *Let $u_0 \in Y$ that satisfies Hypothesis 1. If for some $0 < \gamma < 1$ it holds*

$$\|u_0 - \varphi_c\|_{\mathcal{H}} \le \gamma$$

then

$$|E(u_0) - E(\varphi_c)| \le 2\gamma(2+c) \tag{114}$$

and

$$|F(u_0) - F(\varphi_c)| \le 6\gamma(2+c)^2 \ , \tag{115}$$

where φ_c, ρ_c are defined in (3), (9).

Proof. By the triangle inequality and (5),

$$\begin{aligned} |E(u_0) - E(\varphi_c)| &\le \|u_0 - \varphi_c\|_{\mathcal{H}}(\|u_0\|_{\mathcal{H}} + \|\varphi_c\|_{\mathcal{H}}) \\ &\le \|u_0 - \varphi_c\|_{\mathcal{H}}(\|u_0 - \varphi_c\|_{\mathcal{H}} + 2\|\varphi_c\|_{\mathcal{H}}) \\ &\le \gamma(\gamma + \frac{2}{\sqrt{3}}c) \end{aligned}$$

Now,

$$|F(u_0)-F(\varphi_c)| \le \|u_0-\varphi_c\|_{L^2}\Big\|u^2+2u\varphi_c+\varphi_c^2\Big\|_{L^2}$$
$$\le \|u_0-\varphi_c\|_{L^2}\Big[\|u_0\|_{L^\infty}\|u_0\|_{L^2}+\|\varphi_c\|_{L^\infty}(2\|u_0\|_{L^2}+\|\varphi_c\|_{L^2})\Big]$$

and Lemma 2 together with (10) then yield

$$|F(u_0)-F(\varphi_c)| \le 2\gamma\Big[4\sqrt{\gamma}(2+c)+c(4\gamma+3c)\Big] .$$

□

According to (102)-(103) and Lemma 9, setting $M = v(\xi(T))$, we get

$$M^3-\frac{1}{4}E(u)M+\frac{1}{72}F(u) \le \frac{(2+c)^2\varepsilon^4}{8} .$$

The conservation of E and F together with Lemma 10 and (96) then lead to

$$\begin{aligned} M^3-\frac{1}{4}E(\varphi_c)M+\frac{1}{72}F(\varphi_c) &\le \frac{1}{4}|E(u_0)-E(\varphi_c)|+\frac{1}{72}|F(u_0)-F(\varphi_c)|+\frac{(2+c)^2\varepsilon^4}{8} \\ &\le \varepsilon^4(2+c)^2 \end{aligned} \tag{116}$$

Now, by (5) and (6) one can check that $E(\varphi_c)=c^2/3$ and $F(\varphi_c)=2c^3/3$, so that (116) becomes

$$\Big(\frac{c}{6}-M\Big)^2\Big(M+\frac{c}{3}\Big) \le \varepsilon^4(2+c)^2 .$$

Finally, since according to (112) $M \ge 0$, we deduce that

$$\Big|\frac{c}{6}-M\Big| \le \sqrt{\frac{3}{c}}\,(2+c)\varepsilon^2$$

which together with Lemma 5 , Lemma 10 and (96) ensure that

$$\|u(T)-\varphi_c(x-\xi(T))\|_{\mathcal{H}}^2 \le \varepsilon^2\Big(4\sqrt{3c}(2+c)+2(2+c)\varepsilon^2\Big) \le 4(2+c)^2\varepsilon^2 .$$

This completes the proof of (101) and thus of (15). Note that (16) then follows by using Lemma 2.

6. Stability of a train of well-ordered antipeakons-peakons

In this section, we generalize the stability result to the sum of well ordered trains of antipeakons-peakons (see fig 2 and fig 3). Let be given N_-+N_+ ordered speeds $\vec{c}=(c_{-N_-},..,c_{-1},c_1,..,c_{N_+})\in\mathbb{R}^{N_-+N_+}$ with

$$c_{-N_-}<..<c_{-1}<0<c_1<..<c_{N_+} . \tag{117}$$

We set

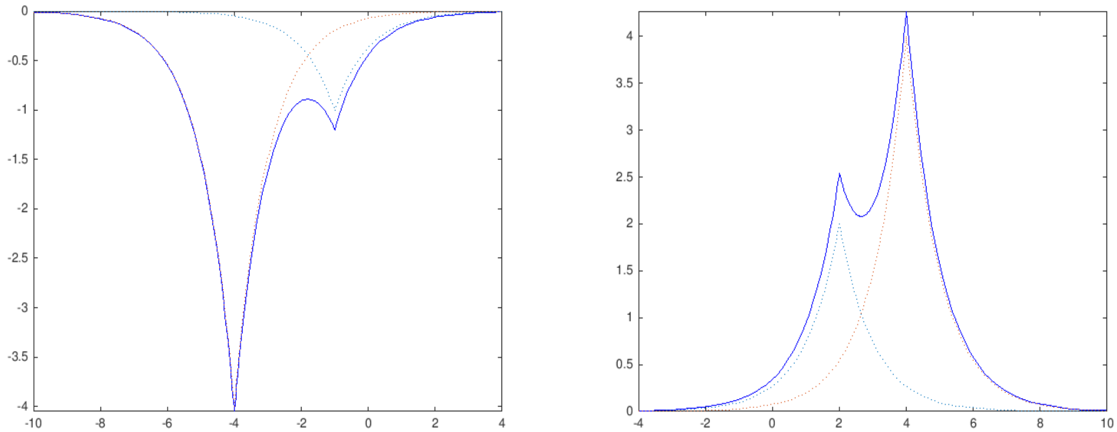
$$\|\vec{c}\|_1=\sum_{j=-N_-}^{N_+}|c_j| \quad\text{and}\quad \sigma(\vec{c})=\min_{i\in[[1-N_-,N_+]]}|c_i-c_{i-1}| \tag{118}$$

where to simplify the notations we set

$$c_0=0 . \tag{119}$$

For $\alpha>0$ and $L>0$ and $\vec{c}$ satisfying (117)-(118), we define the following neighborhood of all the sums of N_-+N_+ well-ordered antipeakons and peakons of speed $c_{-N_-},..,c_{-1},c_1,..,c_{N_+}$ with spatial shifts z_j that satisfied $z_j-z_q\ge L$ for $j>q$.

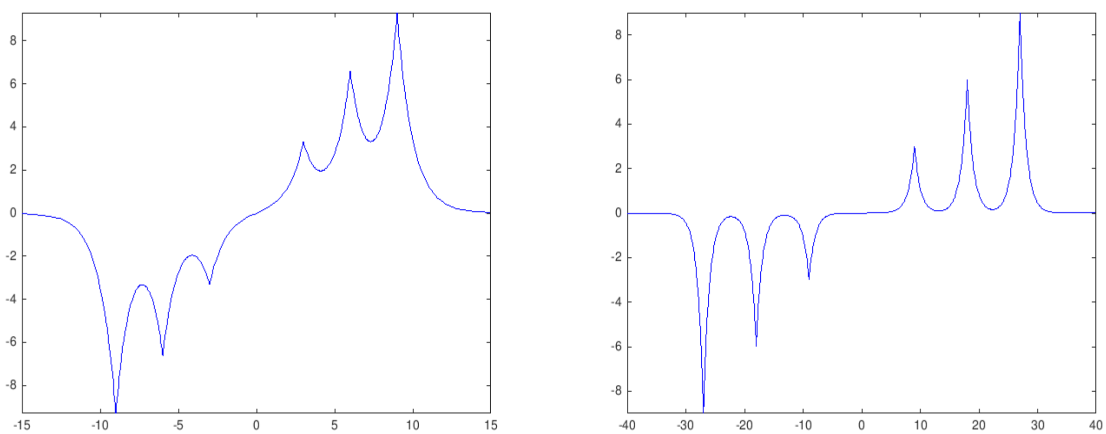
$$U(\alpha,L,\vec{c})=\Big\{u\in L^2(\mathbb{R}),\ \inf_{z_j-z_q>L,\ j>q}\Big\|u-\sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+}\varphi_{c_j}(\cdot-z_j)\Big\|_{\mathcal{H}}<\alpha\Big\} . \tag{120}$$



(A) Two antipeakons at speeds $c_i = 1, 4$.

(B) Two peakons at speeds $c_i = 2, 4$.

FIGURE 2. Summing two antipeakons and peakons profiles at time $t = 1$ with different speeds.



(A) At time $t = 1$.

(B) At time $t = 3$.

FIGURE 3. Three well-ordered trains of antipeakons and peakons profiles at different speeds $c_i = 3, 6, 9$.

We start by establishing the following lemma that linked the distance in L^∞ to the train of antipeakons-peakons with the distance in $\mathcal{H}$. Indeed, applying Lemma 2 with $\psi = \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j)$ and observing that

$$\|\psi\|_{L^\infty} + \|\psi'\|_{L^\infty} \leq 2 \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \|\varphi_{c_j}\|_{L^\infty} \leq 2\|\vec{c}\|_1 \ ,$$

we get the following lemma.

Lemma 11 (L^∞ approximations)**.** *Let $(c_j, z_j) \in \mathbb{R}^2$, $j \in [[N_-, N_+]] \setminus \{0\}$, and $u \in Y$, satisfying Hypothesis 1, then*

$$\Big\| u - \sum_{\substack{j=-N_- \\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j)\Big\|_{L^\infty(\mathbb{R})} \le 2\Big\| u - \sum_{\substack{j=-N_- \\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j)\Big\|_{\mathcal{H}}^{2/3} \Big(1+\sqrt{2}\Big\| u - \sum_{\substack{j=-N_- \\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j)\Big\|_{\mathcal{H}}^{2/3} + 2\|\vec{c}\|_1\Big) . \tag{121}$$

In particular, if moreover $\| u - \sum_{\substack{j=-N_- \\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j)\|_{\mathcal{H}} \le 1/2$ then

$$\Big\| u - \sum_{\substack{j=-N_- \\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j)\Big\|_{L^\infty(\mathbb{R})} \le 4(1+\|\vec{c}\|_1)\Big\| u - \sum_{\substack{j=-N_- \\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j)\Big\|_{\mathcal{H}}^{2/3} . \tag{122}$$

6.1. **Control of the distance between the peakons.** In this subsection we want to prove that for a given $\vec{c}$ satisfying (117) , there exists $\alpha = \alpha(\vec{c})$ and $L = L(\vec{c})$ such that as soon as the solution $u(t)$ stays in $U(\alpha, L, \vec{c})$ the different bumps of u that are individually close to a peakon or an antipeakon get away from each others as time is increasing. This is crucial in our analysis since we do not know how to manage strong interactions.

Lemma 12. *(Decomposition of the solution around a sum of antipeakons and peakons). Let $u_0 \in Y$ satisfying (17)-(19). There exist $\alpha_0(\vec{c}) > 0$, $L_0(\vec{c}) > 0$ and $\tilde{K}(\vec{c}) \ge 1$ such that for all $0 < L_0 < L$ if for some $T > 0$*

$$u \in U(\alpha_0, L/2, \vec{c}) \quad \text{on } [0,T] \tag{123}$$

then there exist $N_- + N_+$ C^1-functions $x_{-N_-}(\cdot) < .. < x_{-1}(\cdot) < x_1(\cdot) < .. < x_{N_+}(\cdot)$ defined on $[0,T]$ such that for all $t \in [0,T]$ we have,

$$\int_{\mathbb{R}} \Big(v(t,x) - \sum_{\substack{j=-N_- \\ j\neq 0}}^{N_+} \rho_{c_j}\big(x - x_j(t)\big)\Big)\partial_x \rho_{c_i}\big(x - x_i(t)\big) dx = 0, \qquad \forall i \in [[-N_-, N_+]], \tag{124}$$

$$|\dot{x}_i(t) - c_i| \le \frac{\sigma(\vec{c})}{8}, \qquad \forall i \in [[-N_-, N_+]]\backslash\{0\}, \tag{125}$$

and

$$x_i(t) - x_j(t) \ge 3L/4, \quad \forall (i,j) \in ([-N_-, N_+]] \setminus \{0\})^2, i > j, \tag{126}$$

where $v = (4-\partial_x^2)^{-1}u$ and $\rho_{c_i} = (4-\partial_x^2)^{-1}\varphi_{c_i}$.
Moreover, if

$$u \in U(\alpha, L/2, \vec{c}) \quad \text{on } [0, t_0] \tag{127}$$

for some $0 < \alpha < \alpha_0(\vec{c})$, then

$$\left\| u(t,\cdot) - \sum_{\substack{i=-N_- \\ i\neq 0}}^{N_+} \varphi_{c_i}(\cdot - x_i(t)) \right\|_{\mathcal{H}} \le \tilde{K}\alpha, \tag{128}$$

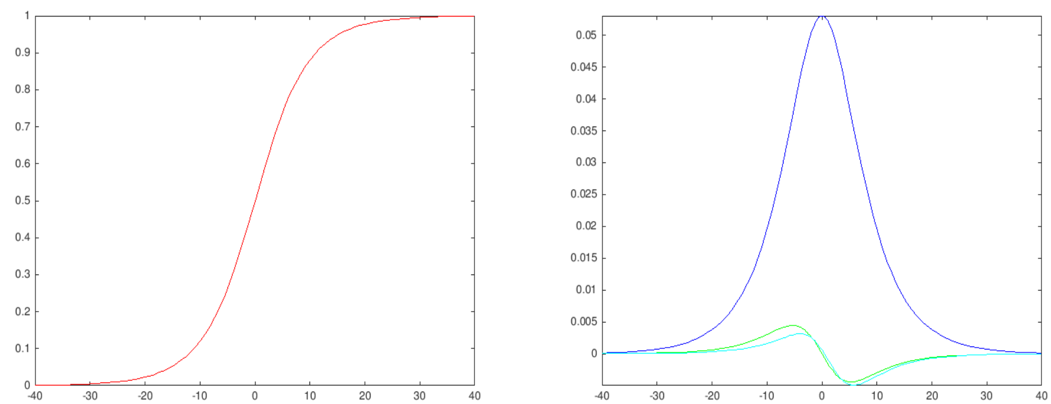
$$\left\| v(t,\cdot) - \sum_{\substack{i=-N_- \\ i\neq 0}}^{N_+} \rho_{c_i}(\cdot - x_i(t)) \right\|_{C^1(\mathbb{R})} \le \tilde{K}\alpha, \tag{129}$$

Proof. The strategy is to use a modulation argument to construct $N_- + N_+$ C^1-functions $t \mapsto x_i(t)$, $i \in [[-N_-, N_+]]\backslash\{0\}$ on $[0,T]$ satisfying the orthogonality conditions (124). The proofs of the above estimates are direct adaptations of similar estimates proved in Lemma 4. We refer to [10, 8] for details. □

6.2. **Monotonicity property.** Thanks to the preceding lemma, for $\alpha_0 > 0$ small enough and $L_0 > 0$ large enough, one can construct C^1-functions $x_{-N_-}(\cdot) < ... < x_{N_+}(\cdot)$ defined on $[0, T]$ such that (128), (129), (125) are satisfied. In this subsection we state the almost monotonicity of functionals that correspond to the part of the functional $E(\cdot) - \lambda F(\cdot)$ at the right of a curve that travels slightly at the left of the ith bump of u. To control the growth of the mass of $y(t)$ we will also need an almost monotonicity result on $E(\cdot) + \gamma M(\cdot)$ at the right of a curve that travels slightly at the left of the smallest positive bump of u. As in [16], we introduce the C^∞-function Ψ defined on $\mathbb{R}$ by

$$\Psi(x) = \frac{2}{\pi} \arctan\Big(\exp(x/6)\Big) \tag{130}$$

It is easy to check that $\Psi(-\cdot) = 1 - \Psi$ on $\mathbb{R}$, Ψ' is a positive even function and that there exists $C > 0$ such



(A) Representative curve of Ψ (B) Representative curves of Ψ',Ψ'',Ψ'''

FIGURE 4. Profiles of Ψ and its derivatives.

that $\forall x \leq 0$,

$$|\Psi(x)| + |\Psi'(x)| \leq C \exp(x/6)\ . \tag{131}$$

Moreover, by direct calculations (see fig 5), it is easy to check that

$$|\Psi^{'''}| \leq \frac{1}{2} \Psi' \text{ on } \mathbb{R}, \tag{132}$$

and that

$$\Psi'(x) \geq \Psi'(2) = \frac{1}{3\pi} \frac{e^{1/3}}{1 + e^{2/3}}, \quad \forall x \in [0, 2]\ . \tag{133}$$

Setting $\Psi_K = \Psi(\cdot/K)$, we introduce for $j \in \{1, .., N_+\}$ and $\lambda \geq 0$,

$$\mathcal{J}_{j,\lambda}(t) = \mathcal{J}_{j,\lambda,K}\big(t, u(t,x)\big) = \int_{\mathbb{R}} \Big(\big[4v^2(t,x) + 5v_x^2(t,x) + v_{xx}^2(t,x)\big] - \lambda u^3(t,x) \Big) \Psi_{j,K}(t)\, dx\,, \tag{134}$$

where $\Psi_{j,K}(t,x) = \Psi_K(x - y_j(t))$ with $y_j(t)$, $j = 1, .., N_+$, defined by

$$y_1(t) = x_1(0) + \frac{c_1}{2} t - \frac{L}{4}, \tag{135}$$

and

$$y_i(t) = \frac{x_{i-1}(t) + x_i(t)}{2}, \quad i = 2, .., N_+. \tag{136}$$

Proposition 4. *(Almost monotony of the functional energy $\mathcal{J}_{i,\lambda,K}$) Let $T > 0$ and $u \in C(\mathbb{R}_+; H^1)$ be the solution of the DP equation emanating from $u_0 \in Y$, satisfying Hypothesis 1 with* (17)-(18) *on* $[0, T]$*. There*

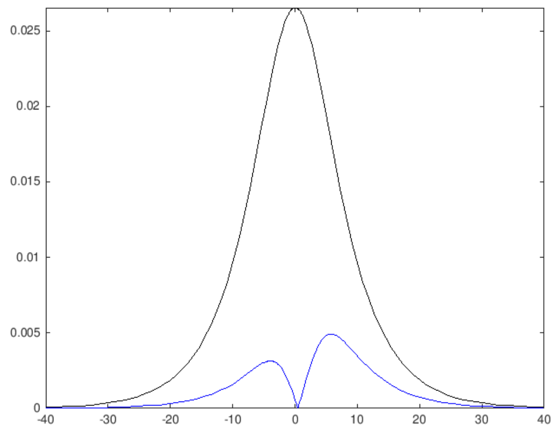


FIGURE 5. Profiles of $|\Psi'''|$ (blue) with respect to $\frac{1}{2}\Psi'$ (black).

exist $\alpha_0(\vec{c}) > 0$ *and* $L_0(\vec{c}) > 0$ *such that if* $0 < \alpha < \alpha_0(\vec{c}) \ll 1$ *and* $L \geq L_0 > 0$ *then for any* $1 \leq K \lesssim \sqrt{L}$ *and* $0 \leq \lambda \leq \frac{1}{2c_1}$,

$$\mathcal{J}_{j,\lambda,K}(t) - \mathcal{J}_{j,\lambda,K}(0) \leq O(e^{-\frac{L}{48K}}), \quad \forall j \in \{1, ..., N_+\}, \quad \forall t \in [0, T] \ . \tag{137}$$

The proof of this proposition relies on the following virial type identities that are proven in the appendix.

Lemma 13. *(Viral type identity). Let* $u \in C(\mathbb{R}_+; H^\infty(\mathbb{R}))$ *be a solution of equation* (4)*. For any smooth function* $g\colon \mathbb{R} \mapsto \mathbb{R}$*, it holds*

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}(4v^2 + 5v_x^2 + v_{xx}^2)(t,x)g\,dx = \frac{2}{3}\int_{\mathbb{R}} u^3(t,x)g'dx - 4\int_{\mathbb{R}} u^2(t,x)v(t,x)g'dx \\ + 5\int_{\mathbb{R}} v(t,x)h(t,x)g'dx + \int_{\mathbb{R}} v_x(t)h_x(t,x)g'dx,\end{aligned} \tag{138}$$

$$\frac{d}{dt}\int_{\mathbb{R}} u^3(t,x)g\,dx = \frac{3}{4}\int_{\mathbb{R}} u^4(t,x)g'dx + \frac{9}{4}\int_{\mathbb{R}}(h^2 - h_x^2)(t,x)g'dx, \tag{139}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}} yg\,dx = \int_{\mathbb{R}} yug'\,dx + \frac{3}{2}\int_{\mathbb{R}}(u^2 - u_x^2)g'\,dx \tag{140}$$

where $y = (1-\partial_x^2)u$, $v = (4-\partial_x^2)^{-1}u$, *and* $h = (1-\partial_x^2)^{-1}u^2$.

Proof of Proposition 4 We first note that combining (136) and (125), it holds for $j = 1, ..., N_+$,

$$\frac{3}{2}c_{N_+} \geq \dot{y}_j(t) \geq \frac{c_1}{2}. \tag{141}$$

Now, using (134), (138) and (139) with $g = \Psi_{j,\lambda,K}(\cdot - y_j(t))$, $j \geq 1$, one gets

$$\begin{aligned}\frac{d}{dt}\mathcal{J}_{j,\lambda,K}(t) &= -\dot{y}_j(t)\int_{\mathbb{R}}(4v^2 + 5v_x^2 + v_{xx}^2)\Psi'_{j,K}(x - y_j(t))dx + \int_{\mathbb{R}}(\frac{2}{3}u - 4v)u^2\Psi'_{j,K}dx \\ &\quad + \int_{\mathbb{R}}(5vh + v_xh_x)\Psi'_{j,K}dx + \lambda\dot{y}_j(t)\int_{\mathbb{R}} u^3\Psi'_{j,K}dx \\ &\quad - \frac{3}{4}\lambda\int_{\mathbb{R}} u^4\Psi'_{j,K}dx - \frac{9}{4}\lambda\int_{\mathbb{R}}(h^2 - h_x^2)\Psi'_{j,K}dx \end{aligned} \tag{142}$$

$$= -\dot{y}_j(t)\int_{\mathbb{R}}(4v^2 + 5v_x^2 + v_{xx}^2)\Psi'_{j,K}(x - y_j(t))dx + F_1 + F_2 + ... + F_5. \tag{143}$$

We claim that for $k=1,2,3$, it holds

$$F_k \le \frac{c_1}{10}\int_{\mathbb{R}}(4v^2+5v_x^2+v_{xx}^2)\Psi'_{j,K}(x-y_j(t))dx+\frac{C}{K}e^{-\frac{1}{6K}(\sigma(\vec{c})t+L/8)}. \tag{144}$$

For all $t\in[0,T]$ and each $j\in[[1,N_+]]$ divide $\mathbb{R}$ into two regions $D_j=D_j(t)$ and D_j^c with

$$D_j(t)=[x_{j-1}(t)+L/4,x_j(t)-L/4] \quad \text{for} \quad j\in[[2,N_+]] \quad \text{and} \quad D_1(t)=[x_1(0)-L/2,x_1(t)-L/4]\ .$$

First, in view of (125) and (135)-(136), one can check that for $x\in D_j^c(t)$, we have

$$|x-y_j(t)|\ge \sigma(\vec{c})\,t+L/8, \tag{145}$$

with $\sigma(\vec{c})$ defined in (118). Indeed, for $j\in[[2,N_+]]$ it holds

$$|x-y_j(t)|\ge \frac{x_j(t)-x_{j-1}(t)}{2}-L/4\ge \frac{c_j-c_{j-1}}{4}\,t+L/8\ge \sigma(\vec{c})t+L/8.$$

and for $j=1$,

$$|x-y_1(t)|\ge \frac{c_1}{4}t+L/4\ge \sigma(\vec{c})t+L/8.$$

Second, noticing that

$$u^2=(4v-v_{xx})^2\le 20v^2+5v_{xx}^2\le 5(4v^2+5v_{xx}^2+v_{xx}^2), \tag{146}$$

and proceeding as for the estimate (43) with the help of (128)-(129) and the exponential decay of φ_{c_j} on D_j, it holds

$$\begin{aligned}
\|v(t,\cdot)\|_{C^1(D_j)}+\|u(t,\cdot)\|_{L^\infty(D_j)} \ &\lesssim\ \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+}\|\varphi_{c_j}(\cdot-x_j(t))\|_{L^\infty(D_j)}+\|u-\sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+}\varphi_{c_j}(\cdot-x_j(t))\|_{L^\infty(D_j)}\\
&+\ \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+}\|\rho_{c_j}(\cdot-x_j(t))\|_{L^\infty(D_j)}+\|v-\sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+}\rho_{c_j}(\cdot-x_j(t))\|_{L^\infty(D_j)}\\
&\le\ O(e^{-L/8})+O(\sqrt{\alpha})\ .
\end{aligned} \tag{147}$$

Now to estimate F_1, we note that combining (145)-(147) and the exponential decay of $\Psi'_{j,K}$ on D_j^c, we get

$$\begin{aligned}
F_1 &\le 4(\|u\|_{L^\infty(D_j)}+\|v\|_{L^\infty(D_j)})\int_{\mathbb{R}}u^2\Psi'_{j,K}dx+4(\|u\|_{L^\infty(D_j^c)}+\|v\|_{L^\infty(D_j^c)})\|\Psi'_{j,K}\|_{L^\infty(D_j^c)}\|u\|^2_{L^2(\mathbb{R})}\\
&\le 20(\|u\|_{L^\infty(D_j)}+\|v\|_{L^\infty(D_j)})\int_{\mathbb{R}}(4v^2+5v_x^2+v_{xx}^2)\Psi'_{j,K}dx+\frac{20}{K}\|u_0\|^3_{\mathcal{H}}e^{-\frac{1}{6K}(\sigma(\vec{c})t+L/8)},
\end{aligned}$$

where we used (39) and that, thanks to (5) and (10),

$$\|v\|_{L^\infty(\mathbb{R})}\le \frac{1}{\sqrt{2}}\|v\|_{H^1}\le \frac{1}{2\sqrt{2}}\|u\|_{\mathcal{H}}=\frac{1}{2\sqrt{2}}\|u_0\|_{\mathcal{H}}\ .$$

Therefore, for $0<\alpha<\alpha_0(\vec{c})\ll 1$ small enough and $L>L_0>0$ large enough, it holds

$$F_1\le \frac{c_1}{10}\int_{\mathbb{R}}(4v^2+5v_x^2+v_{xx}^2)\Psi'_{j,K}(x-y_j(t))dx+\frac{C}{K}e^{-\frac{1}{6K}(\sigma(\vec{c})t+L/8)}.$$

Let us now tackle the estimate of F_2. We first remark that from the definition of Ψ in Section 6.2, and in particular (132), we have for $K\ge 1$,

$$(1-\partial_x^2)\Psi'_{j,K}\ge (1-\frac{1}{2K^2})\Psi'_{j,K}\Rightarrow (1-\partial_x^2)^{-1}\Psi'_{j,K}\le \big(1-\frac{1}{2K^2}\big)^{-1}\Psi'_{j,K} \tag{148}$$

and, by Young's convolution estimates and (10),

$$\|h\|_{L^2}\le \frac{1}{2}\|e^{-|\cdot|}\|_{L^2}\|u^2\|_{L^1}\le \frac{1}{2}\|u\|^2_{L^2}\le 2\|u\|^2_{\mathcal{H}}\ . \tag{149}$$

We also notice that

$$h(x)\ =\ \frac{1}{2}e^{-x}\int_{-\infty}^{x}e^{x'}u^2(x')\,dx'+\frac{1}{2}e^{x}\int_{-\infty}^{x}e^{-x'}u^2(x')\,dx',$$

and

$$h_x(x) = -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{x'}u^2(x')\,dx' + \frac{1}{2}e^{x}\int_{-\infty}^{x} e^{-x'}u^2(x')\,dx',$$

so that

$$|h_x(x)| \le h(x) \quad \forall x \in \mathbb{R}. \tag{150}$$

and thus

$$F_2 \le \int_{\mathbb{R}} (5v + |v_x|)h\Psi'_{j,K} \quad . \tag{151}$$

Therefore, according to (146)-(147) and (148)-(151), we have

$$\begin{aligned} F_2 &\le 6\|v\|_{C^1(D_j)} \int_{\mathbb{R}} h\Psi'_{j,K}dx + \|\Psi'_{j,K}\|_{L^\infty(D_j^c)}\|h\|_{L^2}(5\|v\|_{L^2} + \|v_x\|_{L^2}) \\ &\le 6\|v\|_{C^1(D_j)} \int_{\mathbb{R}} u^2(1-\partial_x^2)^{-1}\Psi'_{j,K}dx + 2\|\Psi'_{j,K}\|_{L^\infty(D_j^c)}\|u\|_{\mathcal{H}}^2(5\|v\|_{L^2} + \|v_x\|_{L^2}) \\ &\le 30\|v\|_{C^1(D_j)}\big(1-\frac{1}{2K^2}\big)^{-1}\int_{\mathbb{R}}(4v^2+5v_x^2+v_{xx}^2)\Psi'_{j,K}dx + 6\|\Psi'_{j,K}\|_{L^\infty(D_j^c)}\|u\|_{\mathcal{H}}^3 \ . \end{aligned}$$

Using (10) and the exponential decay of of $\Psi'_{j,K}$ on D_j^c with (145), we thus get

$$F_2 \le 30\big(1-\frac{1}{2K^2}\big)^{-1}\|v\|_{C^1(D_j)}\int_{\mathbb{R}}(4v^2+5v_x^2+v_{xx}^2)\Psi'_{j,K}dx + 6\|u_0\|_{\mathcal{H}}^3 e^{-\frac{1}{6K}(\sigma(\vec{c})+L/8)},$$

so that by (151) F_2 satisfies (144) for $0 < \alpha < \alpha_0(\vec{c}) \ll 1$ small enough and $L > L_0 \gg 1$ large enough.

To estimate F_3, remark that using (141) and (146) one may write

$$F_3 \le \frac{15}{2}\lambda c_{N_+}\|u\|_{L^\infty(D_j)}\int_{\mathbb{R}}(4v^2+5v_x^2+v_{xx}^2)\Psi'_{j,K}dx + \frac{3}{2}\lambda c_{N_+}\|\Psi'_{j,K}\|_{L^\infty(D_j^c)}\|u\|_{L^\infty(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}^2 \ .$$

Using that, by hypothesis $0 \le \lambda \le (2c_1)^{-1}$, the exponential decay of $\Psi'_{j,K}$ on D_j^c, (10) and (39), we deduce that F_3 satisfies (144) for $0 < \alpha < \alpha_0(\vec{c}) \ll 1$ small enough and $L > L_0(\vec{c}) \gg 1$ large enough.

Finally, $\Psi'_{j,K} \ge 0$, $\lambda \ge 0$ and (150) ensure that $F_4 + F_5$ is non positive. Gathering (141)-(144) we thus infer that

$$\frac{d}{dt}\mathcal{J}_{j,\lambda,K}(t) \le \frac{C}{K}\|u_0\|_{\mathcal{H}}^3 e^{-\frac{1}{6K}(\sigma(\vec{c})t+L/8)}.$$

Integrating this inequality between 0 and t, (137) follows and this proves the proposition for smooth initial solutions. Finally, approximating the initial data as in (25), the strong continuity result with respect to initial data (33) in Proposition 2 ensures that (137) also hold for $u_0 \in Y$ satisfying Hypothesis 1. □

We will also need the following monotonicity result on $E + \gamma M$ at the right of the curve $y_1(\cdot)$. We introduce the function ϕ defined by

$$\Phi(x) = \begin{cases} 0 & \text{for} \quad x \le 0 \\ x/2 & \text{for} \quad x \in [0,2] \\ 1 & \text{for} \quad x \ge 2 \end{cases} \tag{152}$$

Lemma 14. *Let $u \in C([0,T]; H^\infty \cap L^\infty(0,T;Y)$ be the solution to (4) satisfying Hypothesis 1 and (123) for some $L > 0$. Assume moreover that u satisfies*

$$x_0(t) \le x_1(0) - L/4 + \frac{c_1 t}{2}\ , \quad \forall t \in [0,T], \tag{153}$$

where $x_1(\cdot)$ is defined in Lemma 12. There exists $L_0 = L_0(\vec{c}) > 0$ such that if $L \ge L_0$ then on $[0,T]$, it holds

(154)

$$\int_{\mathbb{R}}\Big(4v^2+5v_x^2+v_{xx}^2\Big)(t)\Psi(\cdot-y_1(t))+\frac{c_1}{2^9}y\Phi(\cdot-y_1(t)) \le \int_{\mathbb{R}}\Big(4v_0^2+5v_{0,x}^2+v_{0,xx}^2\Big)\Psi(\cdot-y_1(0))+\frac{c_1}{2^9}y(0)\Phi(\cdot-y_1(0))+O(e^{-\frac{L}{48}})$$

where $y_1(\cdot)$ is defined in (135) and Ψ is defined in (130).

Proof. Applying (138) with $g(t,x) = \Psi(x - y_1(t))$ and (140) with $g(t,x) = \Phi(x - y_1(t))$ and recalling the definition (135) of $y_1(\cdot)$, we get

$$\frac{d}{dt}\big[E(u) + \frac{c_1}{2^9}M(u)\big] = -\frac{c_1}{2}\int_{\mathbb{R}}\Big[\Psi'(4v^2 + 5v_x^2 + v_{xx}^2) + \frac{c_1}{2^9}\Phi' y\Big] + \frac{3c_1}{2^{10}}\int_{\mathbb{R}}(u^2 - u_x^2)\phi' + \frac{c_1}{2^9}\int_{\mathbb{R}} uy\Phi' + J \tag{155}$$

where thanks to (144),

$$\begin{aligned} J &= \int_{\mathbb{R}}(\frac{2}{3}u - 4v)u^2\Psi' dx + \int_{\mathbb{R}}(5vh + v_x h_x)\Psi' dx \\ &\le \frac{c_1}{2^4}\int_{\mathbb{R}}(4v^2 + 5v_x^2 + v_{xx}^2)\Psi' + C\,\|u_0\|_{\mathcal{H}}^3 e^{-\frac{1}{6}(\sigma(\vec{c}) + L/8)} . \end{aligned} \tag{156}$$

We first observe that

$$\int_{\mathbb{R}}(u^2 - u_x^2)\Phi' \le \int_{\mathbb{R}} u^2\Phi' = \int_{\mathbb{R}}(4v - v_{xx})^2\Phi' \le 5\int_{\mathbb{R}}\Big(4v^2 + 5v_x^2 + v_{xx}^2\Big)\Phi' , \tag{157}$$

where, according to the definition (152) of Φ, it holds

$$\frac{3c_1}{2^{10}}\,5\Phi' \le \frac{c_1}{4}\Psi' \quad \text{ on } \mathbb{R} . \tag{158}$$

Second, (153) together with (135) and the definition (152) of Φ ensure that $y(t,\cdot)$ is non negative on the support of $\Phi'(\cdot - y_1(t))$ that is $[y_1(t), y_1(t) + 2]$. Therefore (147) leads to

$$\frac{c_1}{2^9}\int_{\mathbb{R}} uy\Phi' \quad \le \quad \frac{c_1}{2^9}\|u\|_{L^\infty(]y_1(t),y_1(t)+2[)}\int_{\mathbb{R}} y\Phi' dx \le \frac{c_1}{2^{11}}\int_{\mathbb{R}} y\Phi'\, dx . \tag{159}$$

Therefore (157)-(159) and(128) we obtain

$$-\dot{y}_1(t)\int_{\mathbb{R}}\Big[\Psi'(4v^2 + 5v_x^2 + v_{xx}^2) + \frac{c_1}{2^9}\Phi' y\Big] + \frac{3c_1}{2^{10}}\int_{\mathbb{R}}(u^2 - u_x^2)\Phi' + \frac{c_1}{2^9}\int_{\mathbb{R}} uy\Phi' \le -\frac{c_1}{4}\int_{\mathbb{R}}\Psi'(4v^2 + 5v_x^2 + v_{xx}^2)dx$$

that leads to

$$\frac{d}{dt}\big[E(u) + \frac{c_1}{2^9}M(u)\big] \le C\,\|u_0\|_{\mathcal{H}}^3 e^{-\frac{1}{6}(\sigma(\vec{c}) + L/8)} .$$

This proves (154) by integrating in time. □

6.3. **Control of the growth of $\|y\|_{L^1}$.** The control of the growth of the mass of $y(t)$ is more delicate than in the case of the stability of a single peakon. Indeed, in this last case we deeply use that u stays L^∞-close to the peakon that is positive and thus the negative part of u stays small. In the present case, this is of course no more true because our train of antipeakon-peakons is no more positive. To overcome this difficulty we make use of the monotony argument for $E(u) + \gamma M(u)$ proven in Lemma 14.

Proposition 5. *Let $u_0 \in Y \cap H^\infty(\mathbb{R})$ satisfying Hypothesis 1 and $u \in C(\mathbb{R}_+; H^\infty) \cap L^\infty_{loc}(\mathbb{R}_+; Y)$ be the associated solution to DP given by Proposition 2. There exist $\alpha_0 = \alpha_0(\vec{c})$ and $L_0 = L_0(\vec{c})$ such that if*

$$u(t) \in U(\alpha, L, \vec{c}) \quad , \quad \forall t \in [0,T] \tag{160}$$

with $0 < \alpha \le \alpha_0$ and $L \ge L_0$ then

$$\|y(t,\cdot)\|_{L^1(\mathbb{R})} \le e^{17 + 2^{-5}(c_1 \wedge |c_{-1}|)t}\frac{(\|\vec{c}\|_1 + 1)^3}{(c_1 \wedge |c_{-1}|)^2}(1 + \|y_0\|_{L^1}) \quad , \quad \forall t \in [0,T]. \tag{161}$$

Proof. In view of Lemma 12, there exists $N_- + N_+$ C^1-functions $x_{-N_-}(\cdot) < .. < x_{-1}(\cdot) < x_1(\cdot) < .. < x_{N_+}(\cdot)$ defined on $[0,T]$ that satisfy (125)-(126) and (128)-(129).

We separate two cases depending on the place of $x_0(0)$ with respect to $x_1(0)$.

Case 1. $x_0(0) \le x_1(0) - L/3$. Then according to (125), (128), the definition (32) of $x_0(\cdot)$ and a continuity argument, $x_0(t) \le x_1(t) - L/3$ and in particular $\dot{x}_0(t) \le c_1/2$ for all $t \in [0,T]$. This ensures that

$$x_0(t) + L/12 \le y_1(t) = x_1(0) - L/4 + \frac{c_1}{2}t, \quad \forall t \in [0,T],$$

where $y_1(\cdot)$ is defined in (135).
Therefore Lemma 14 leads to

$$\int_{\mathbb{R}}(4v^2+5v_x^2+v_{xx}^2)\Psi(\cdot-y_1(t))+\frac{c_1}{2^9}\int_{\mathbb{R}} y\phi(\cdot-y_1(t)) \le \int_{\mathbb{R}}(4v^2+5v_x^2+v_{xx}^2)\Psi(\cdot-y_1(0))+\frac{c_1}{2^9}\int_{\mathbb{R}} y(0)\phi(\cdot-y_1(0))+O(e^{-\frac{L_0}{48}})$$

Making use of the conservation of E and of the definition of Ψ, if follows that for L large enough,

$$\int_{y_1(t)+2}^{+\infty} y(t,x)\,dx \le \frac{2^9}{c_1}E(u_0)+\|y_0\|_{L^1}+O(e^{-\frac{L_0}{48}}) \le 1+\frac{2^9}{c_1}E(u_0)+\|y_0\|_{L^1}\ , \quad \forall t\in[0,T]\ .$$

On the other hand, according to (28), $u_x \ge -u$ on $]x_0(t),+\infty[$ and by Lemma 11 $\forall t\in[0,T]$ we have,

$$u(t) \le \sum_{i=1}^{N_+} c_i+O(\sqrt{\alpha_0}) \quad \text{on} \quad [y_1(t)+2,+\infty] \quad \text{and} \quad u(t)\le O(\sqrt{\alpha_0})+O(e^{-L_0/8}) \le O(\sqrt{\alpha_0}) \quad \text{on} \quad [x_0(t),y_1(t)+2]\ ,$$

where to get the last inequality we take $L_0>0$ such that $O(e^{-L_0/8})\le\sqrt{\alpha_0}$. Therefore, according to (7) and (38), we have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} y^+(t,x)dx &= \frac{d}{dt}\int_{q(t,x_0)}^{+\infty} y(t,x)dx = -2\int_{x_0(t)}^{+\infty} u_x(t,x)y(t,x)dx\\
&\le 2\int_{x_0(t)}^{y_1(t)+2} u(t,x)y(t,x)dx+2\int_{y_1(t)+2}^{+\infty} u(t,x)y(t,x)dx\\
&\le 2\big(\|\vec{c}\|_1+O(\sqrt{\alpha_0})\big)\big(1+\frac{2^9}{c_1}E(u_0)+\|y_0\|_{L^1}\big)+O(\sqrt{\alpha_0})\int_{\mathbb{R}} y^+(t,x)dx.
\end{aligned}$$

Hence, Grönwall's inequality yields $\forall t\in[0,T]$

$$\int_{\mathbb{R}} y^+(t,x)dx \le e^{C\sqrt{\alpha_0}t}\Big(\|y_0\|_{L^1}+2t\,\big(\|\vec{c}\|_1+1\big)\big(1+\frac{2^9}{c_1}E(u_0)+\|y_0\|_{L^1}\big)\Big), \tag{162}$$

for some universal constant $C>0$. Since, according to Proposition 1, $M(u)=\int_{\mathbb{R}} y$ is conserved for positive times, it follows that

$$\|y(t,\cdot)\|_{L^1(\mathbb{R})} \le 2e^{C\sqrt{\alpha_0}t}\Big(\|y_0\|_{L^1}+2t\,\big(\|\vec{c}\|_1+1\big)\big(1+\frac{2^9}{c_1}E(u_0)+\|y_0\|_{L^1}\big)\Big). \tag{163}$$

Taking $\alpha_0\le(c_1\wedge|c_{-1}|)^2(C\,2^{10})^{-2}$ we thus deduce that

$$\|y(t,\cdot)\|_{L^1(\mathbb{R})} \le 2e^{2^{-10}(c_1\wedge|c_{-1}|)t}\Big(\|y_0\|_{L^1}+2t\big(\|\vec{c}\|_1+1\big)\big(1+\frac{2^9}{c_1}E(u_0)+\|y_0\|_{L^1}\big)\Big)\ .$$

Since for $t\ge0$, $te^{2^{-10}(c_1\wedge|c_{-1}|)t}\le te^{2^{-6}(c_1\wedge|c_{-1}|)t}e^{2^{-5}(c_1\wedge|c_{-1}|)t}\le\frac{e^{2^{-5}(c_1\wedge|c_{-1}|)t}}{2^{-6}(c_1\wedge|c_{-1}|)}e^{-1}$, it follows that

$$\|y(t,\cdot)\|_{L^1(\mathbb{R})} \le 2e^{2^{-5}(c_1\wedge|c_{-1}|)t}\Big(\|y_0\|_{L^1}+2^7\frac{\big(\|\vec{c}\|_1+1\big)}{(c_1\wedge|c_{-1}|)}\big(1+\frac{2^9}{(c_1\wedge|c_{-1}|)}E(u_0)+\|y_0\|_{L^1}\big)\Big)\ . \tag{164}$$

Finally, taking $\alpha_0\le1$, (160) ensures that $E(u_0)\le(\|\vec{c}\|_1+1)^2$, and noticing that

$$\frac{\|\vec{c}\|_1+1}{c_1\wedge|c_{-1}|}\ge1,$$

we eventually get (161).

Case 2: $x_0(0)\ge x_1(0)-L/3$. Then by (126), we must have $x_0(0)\ge x_{-1}(0)+L/3$.

In this case, we make use of the fact that the DP equation is invariant by the change of unknown $u(t,x)\mapsto\tilde{u}(t,x)=-u(t,-x)$. Clearly $\tilde{u}(0)$ also satisfies hypothesis 1 with $\tilde{x}_0(t)=-x_0(t)$. Morever, $\tilde{u}$ satisfies (128) on $[0,T]$ with N_- and N_+ respectively replaced by $\tilde{N}_-=N_+$ and $\tilde{N}_+=N_-$, c_i replaced by $\tilde{c}_i=-c_{-i}$ and $x_i(t)$ replaced by $\tilde{x}_i(t)=-x_{-i}(t)$. In particular, it holds

$$\tilde{x}_0(0)=-x_0(0)\le-x_{-1}-L/3=\tilde{x}_1(0)-L/3,$$

and thus $\tilde{u}$ satisfies the hypothesis of Case 1. Therefore $\tilde{y}=\tilde{u}-\tilde{u}_{xx}=-y(t,-\cdot)$ satisfies (164) with c_{-1} and c_1 respectively replaced by $\tilde{c}_{-1}=-c_1$ and $\tilde{c}_1=-c_{-1}$. This completes the proof of (161) . ☐

Let us now state the adaptation of Proposition 3 in the present case. The role of $x(\cdot)$ will be now play by $x_1(\cdot)$ that localizes the slowest peakon. The proof is essentially the same as the one of Proposition 3. However, in the present case (48) is not available anymore on $\mathbb{R}$ but we actually only need that it holds on $[x_{-1}(t)+L/4,+\infty[$ that is verified since $\sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j} \ge O(\sqrt{\alpha_0})+O(e^{-L_0/8})$ on this interval.

Proposition 6. *There exists $\alpha_0(\vec{c})>0$ and $L_0(\vec{c})>0$ such that for any $u_0\in Y\cap H^\infty(\mathbb{R})$ satisfying Hypothesis 1, if the solution $u\in C(\mathbb{R}_+;H^\infty(\mathbb{R}))$ emanating from u_0 satisfies for some $0<\alpha<\alpha_0$, $L\ge L_0$ and $T>0$,*

$$u\in U\Big(\alpha, L/2,\vec{c}\Big) \qquad on\ [0,T], \tag{165}$$

then for all $t\in[0,T]$,

$$\|y^-(t,\cdot)\|_{L^1(]x_1(t)-\frac{1}{16}c_1t,+\infty[)}\le e^{-c_1t/8}\|y_0\|_{L^1(\mathbb{R})}, \tag{166}$$

where $y^-=\max(-y,0)$, and $x_1(\cdot)$ is the C^1-function constructed in Lemma 12. Moreover it holds

$$u(t,\cdot)-6v(t,\cdot)\le e^{27-\frac{c_1t}{32}}\frac{(\|\vec{c}\|_1+1)^3}{(c_1\wedge|c_{-1}|)^2}(1+\|y_0\|_{L^1}) \quad on\ ]x_1(t)-8,+\infty[\ , \tag{167}$$

where $v=(4-\partial_x)^{-1}u$.

Proof. As mentioned above we mainly proceed as in Proposition 3 but with $x(\cdot)$ replaced by $x_1(\cdot)$. Hence, for $t\in[0,T]$, we separate two possible cases according to the distance between $x_0(t/2)$ and $x_1(t/2)$.
Case 1:

$$x_0(t/2)<x_1(t/2)-\ln(3/2). \tag{168}$$

In this case, the same continuity argument as in the proof of Proposition 3 ensures that

$$x_1(t)-x_0(t)\ge \ln(3/2)+\frac{c_1}{16}t\ . \tag{169}$$

This proves that $y^-(t,\cdot)=0$ on $]x_1(t)-\frac{1}{16}c_1t,+\infty[$ and thus that (166) holds in this case.
Case 2:

$$x_0(t/2)\ge x_1(t/2)-\ln(3/2). \tag{170}$$

Then, as in the proof of Proposition 3, (166) is a consequence of the two following estimates :

$$\left|\int_{x_0(t/2)-\ln 2}^{x_0(t/2)} y(t/2,s)ds\right|\le e^{-\frac14 c_1t}\|y_0\|_{L^1(\mathbb{R})} \tag{171}$$

and

$$\left|\int_{x_1(t)-\ln(3/2)-\frac{c_1}{16}t}^{x_0(t)} y(t,s)\,ds\right|\le e^{c_1t/8}\left|\int_{x_0(t/2)-\ln 2}^{x_0(t/2)} y(t/2,s)\,ds\right|\ . \tag{172}$$

(171) can be obtained exactly as (75) in Proposition 3. We thus focus on (172) where there is the main change. Indeed, we are not allowed to use (48) in order to prove the crucial estimate (81). The idea to overcome this difficulty is to notice that actually we only need such estimate from below on u in $[x_{-1}(t)+L/4,+\infty[$.

Indeed, let q_t be the flow-map defined in (79). For L large enough, (126) and (170) ensure that $x=q_{t/2}(t/2,x)\ge x_{-1}(t/2)+L/2$ as soon as $x\in[x_0(t/2)-\ln 2,x_0(t/2)]$. Therefore, by (125), (129), (122) and a continuity argument, for $\tau\in[t/2,T]$ it holds

$$q_{t/2}\big(\tau,x\big)-x_{-1}(\tau)\ge L/2, \qquad \forall x\in[x_0(t/2)-\ln 2,x_0(t/2)]\ .$$

On the other hand, (129) and (122) ensure that for all $\tau\in[0,T]$,

$$u(\tau,x)\ge -2^{-5}c_1 \quad \text{on} \quad [x_{-1}(\tau)+L/4,+\infty[\ .$$

Combining the two above estimates with (28) we obtain as in Proposition 3 that for any $\tau\in[t/2,t]$ and any $x\in[x_0(t/2)-\ln 2,x_0(t/2)]$,

$$\partial_x q_{t/2}(t,x)\ge exp\Big(-\int_{t/2}^t 2^{-5}c_1\,ds\Big)\ge e^{-2^{-4}c_1t}\ . \tag{173}$$

Once we have the above estimate, the rest of the proof of (166) follows the same lines as in the proof of Proposition 3.

Finally to prove (167), we take α_0 and L_0 that are suitable for Proposition 5 . (87) together with (166) , (161) ensure that for $x \geq x_1(t) - 8$ it holds

$$\begin{aligned} 6v(x) - u(x) &\geq -\frac{1}{2}\int_{-\infty}^{x_1(t)-\frac{c_1}{16}t} e^{-|x-z|} y^-(z)\,dz - \frac{1}{2}\int_{x_1(t)-\frac{c_1}{16}t}^{+\infty} e^{-|x-z|} y^-(z)\,dz \\ &\geq -e^{0\wedge(8-\frac{c_1}{16}t)} e^{2^{-5}(c_1\wedge|c_{-1}|)t}\, e^{17}\,\frac{(\|\vec{c}\|_1+1)^3}{(c_1\wedge|c_{-1}|)^2}(1+\|y_0\|_{L^1}) \\ &\quad -\frac{1}{2}e^{-c_1 t/8}\|y_0\|_{L^1(\mathbb{R})} \\ &\geq -e^{18}\, e^{9-\frac{c_1 t}{32}}\,\frac{(\|\vec{c}\|_1+1)^3}{(c_1\wedge|c_{-1}|)^2}(1+\|y_0\|_{L^1})\,. \end{aligned}$$

□

6.4. **An approximate solution.** A new difficulty with respect to the case of a single peakon will be that

$$t \mapsto \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j^0 - c_j t)$$

is not an exact solution of the DP equation. The aim of the following lemma is to overcome this difficulty by proving that if $L > 0$ is large enough then this is an approximate solution with an error in $L^2(\mathbb{R})$ of order $e^{-L/2}$ on a time interval of order $\ln(L^{3/4})$.

Lemma 15. *Let be given $N_- \in \mathbb{N}^*$ negative velocities $c_{-N_-} < .. < c_{-1} < 0$, $N_+ \in \mathbb{N}^*$ positive velocities $0 < c_1 < .. < c_{N_+}$ and $z_{-N_-}^0 < ..z_{-1}^0 < z_1^0 < .. < z_{N_+}^0$. There exists $L_0 > 0$ only depending on $\vec{c}$ such that for any $L \geq L_0$ if*

$$z_i^0 - z_j^0 \geq L \quad \text{for} \quad i > j \tag{174}$$

then the solution u to (4) emanating from $u_0 = \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j^0)$ satisfies

$$\sup_{t\in[0,2^5(c_1\wedge|c_{-1}|)^{-1}\ln(L^{3/4})]} \Big\| u(t) - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j^0 - c_j t)\Big\|_{\mathcal{H}} \leq e^{-L/2}\,.$$

Proof. We set $\underline{u}(t) = \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j^0 - c_j t)$. Using that $\varphi_c(x-ct)$ is a solution to (4), one can check that $\underline{u}$ satisfies

$$\underline{u}_t + \underline{u}\,\underline{u}_x + \frac{3}{2}\partial_x(1-\partial_x^2)^{-1}(\underline{u}^2) = F \tag{175}$$

with

$$F := \sum_{i<j} c_i c_j \Big(1 + 3(1-\partial_x^2)^{-1}\Big)\partial_x\Big(\varphi(\cdot - z_i^0 - c_i t)\varphi(\cdot - z_j^0 - c_j t)\Big)\,.$$

On account of (174), straightforward calculations lead to

$$\sup_{t\in[0,T]} \Big\|\partial_x\Big(\varphi(\cdot - z_i^0 - c_i t)\varphi(\cdot - z_j^0 - c_j t)\Big)\Big\|_{L^1} + \Big\|\partial_x^2\Big(\varphi(\cdot - z_i^0 - c_i t)\varphi(\cdot - z_j^0 - c_j t)\Big)\Big\|_{\mathcal{M}} \lesssim (L+1)e^{-2L/3}\,.$$

so that

$$\sup_{t\in\mathbb{R}_+} \|F(t)\|_{L^1} + \|F_x(t)\|_{\mathcal{M}} \lesssim (L+1)e^{-2L/3}\,. \tag{176}$$

Note also that for all $t \geq 0$ it holds $\sup_{t\in[0,T]} \|(\underline{u} - \underline{u}_{xx})(t)\|_{\mathcal{M}} = \|\vec{c}\|_1$.

Now, since $\underline{u}(0) = u_0$ clearly satisfies Hypothesis 1, the solution u to (4) emanating from $u_0 = \underline{u}(0)$ exists for all positive times in Y. For $T > 0$ we set

$$M_T = \sup_{t\in[0,T]} \|u - u_{xx}\|_{\mathcal{M}} .$$

At this stage it s worth noticing that Proposition 5 ensures that

$$M_T \le e^{17+2^{-5}(c_1\wedge|c_{-1}|)T} \frac{(\|\vec{c}\|_1+1)^4}{(c_1\wedge|c_{-1}|)^2} . \tag{177}$$

Setting $w = u - \underline{u}$, using exterior regularization and proceeding as in [3] (see also [9] for the DP equation pp: 480-482), we get on $[0,T]$

$$\begin{aligned}\frac{d}{dt}\Big(\int_{\mathbb{R}} |\rho_n * w| + |\rho_n * w_x|\Big) \lesssim &(M_T + \|\vec{c}\|_1)\Big(\int_{\mathbb{R}} |\rho_n * w| + |\rho_n * w_x|\Big) \\ &+ \int_{\mathbb{R}} (|\rho_n * F| + |\rho_n * F_x|) + R_n(t)\end{aligned}$$

where $(\rho_n)_{n\ge0}$ is defined in (24),

$$R_n(t) \to 0 \text{ as } n \to +\infty \text{ and } |R_n(t)| \lesssim 1,\ n \ge 1, t \in \mathbb{R}_+.$$

Therefore Gronwall inequality and since $w(0) = w_x(0) = 0$, yields to

$$\int_{\mathbb{R}} |\rho_n * w(t)| + |\rho_n * w_x(t)| \lesssim \int_0^t e^{C(M_T+\|\vec{c}\|_1)(t-s)}\Big(\int_{\mathbb{R}} |\rho_n * F(s)| + |\rho_n * F_x(s)| + |R_n(s)|\Big) ds . \tag{178}$$

Letting n tends to $+\infty$ and making use of (176) and then (177), we thus get that for L large enough

$$\sup_{t\in[0,T]} \|w(t)\|_{\mathcal{H}} \le \sup_{t\in[0,T]} \|w(t)\|_{L^2} \le \sup_{t\in[0,T]} \|w(t)\|_{W^{1,1}} \le e^{C(1+M_T+\|\vec{c}\|_1)T} e^{-5L/8} . \tag{179}$$

This estimate together with (177) ensure that there exists $L_0(\vec{c}) \ge 1$ such that for all $L > L_0$,

$$\Big\|u(t) - \underline{u}(t)\Big\|_{\mathcal{H}} \le e^{-L/2} \tag{180}$$

as soon as

$$0 \le t \le 2^5(c_1 \wedge |c_{-1}|)^{-1} \ln(L^{3/4}) . \tag{181}$$

Indeed, as soon as (180)-(181) are satisfied, (177) gives

$$M_T \le e^{17} \frac{(\|\vec{c}\|_1+1)^4}{(c_1\wedge|c_{-1}|)^2} L^{3/4}$$

so that (179) leads to

$$\|u(t) - \underline{u}(t)\|_{\mathcal{H}} \le \exp\Big(C_1 + C_2\, L^{3/4} \ln(L^{3/4})\Big) e^{-5L/8} .$$

where $C_1 = C_1(\vec{c}) > 0$ and $C_2 = C_2(\vec{c}) > 0$. This gives (180) for L large enough and proves the result by a continuity argument. □

6.5. **Two global estimates.** The following generalization of the quadratic identity in Lemma 5 was proved in [10].

Lemma 16. *(Global quadratic identity) Let $u \in L^2(\mathbb{R})$ and assume that $z_{-N_-} < .. < z_{-1} < z_1 < .. < z_{N_+}$ with $z_i - z_j \ge L/2$ for $i > j$. Then it holds*

$$E(u) - \sum_{i=-N_-}^{N_+} E(\varphi_{c_i}) = \Big\|u - \sum_{i=-N_-}^{N_+} \varphi_{c_i}(\cdot - z_i)\Big\|_{\mathcal{H}}^2 + 4\sum_{i=-N_-}^{N_+} c_i\Big(v(z_i) - \frac{c_i}{6}\Big) + O(e^{-L/2}) \quad i \in [[-N_-, N_+]]\backslash\{0\}. \tag{182}$$

where $v = (4 - \partial_x^2)^{-1} u$.

Proof. First, according to the definition of the energy space (5) we notice that

$$\Big\|u-\sum_{i=-N_-}^{N_+}\varphi_{c_i}(\cdot-z_i)\Big\|_{\mathcal H}^2 = E(u)+E\Big(\sum_{i=-N_-}^{N_+}\varphi_{c_i}(\cdot-z_i)\Big)-2\sum_{i=-N_-}^{N_+}\big\langle(1-\partial_x^2)\varphi_{c_i}(\cdot-z_i),v\big\rangle$$

$$= E(u)+E\Big(\sum_{i=-N_-}^{N_+}\varphi_{c_i}(\cdot-z_i)\Big)-4\sum_{i=-N_-}^{N_+}c_iv(z_i). \tag{183}$$

where we used that $(1-\partial_x^2)\varphi_{c_i}(\cdot-z_i)=2c_i\delta_{z_i}$ with δ_{z_i} the Dirac mass applied at point z_i. However,

$$E\Big(\sum_{i=-N_-}^{N_+}\varphi_{c_i}(\cdot-z_i)\Big) = \sum_{i=1}^{N}\sum_{j=1}^{N}\langle(1-\partial_x^2)\varphi_{c_i}(\cdot-z_i),\rho_{c_j}(\cdot-z_j)\rangle$$

$$= 2\sum_{i=-N_-}^{N_+}\sum_{j=-N_-}^{N_+}c_i\rho_{c_j}(z_i-z_j)$$

$$= \frac{1}{3}\sum_{i=-N_-}^{N_+}c_i^2+2\sum_{i=-N_-}^{N_+}c_i\sum_{\substack{j=-N_-\\ j\neq i}}^{N_+}\rho_{c_j}(z_i-z_j)\ . \tag{184}$$

From the definition of ρ_{c_j} in (9) and the fact that $z_i-z_j\ge 2L/3$ for $i>j$, it follows that

$$\Big|\sum_{\substack{j=-N_-\\ j\neq i}}^{N_+}\rho_{c_j}(z_i-z_j)\Big| = \Big|\sum_{\substack{j=-N_-\\ j\neq i}}^{N_+}\frac14\Big(e^{-2|\cdot|}*\varphi_{c_j}(\cdot-z_j)\Big)(z_i)\Big| = \Big|\sum_{\substack{j=-N_-\\ j\neq i}}^{N_+}\frac{c_j}{3}e^{-|z_i-z_j|}-\frac{c_j}{6}e^{-|z_i-z_j|}\Big|$$

$$\le \|\vec c\|_1 e^{-2L/3}\le O(e^{-L/2}) \tag{185}$$

Gathering (183), (184), (185) with $E(\varphi_{c_i})=c_i^2/3$ then (182) holds for $L>L_0\gg 1$ large enough. $\square$

The following lemma is an adaptation of Lemma 6 in the present case.

Lemma 17. *Let $u\in L^\infty(\mathbb R)\cap L^2(\mathbb R)$ such that*

$$\Big\|u-\sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+}\varphi_{c_i}(\cdot-z_j)\Big\|_{L^\infty(\mathbb R)}\le \frac{10^{-5}}{N_-+N_+}(c_1\wedge|c_{-1}|) \tag{186}$$

for some $c_{-N_-}<..<c_{-1}<0<c_1<..<c_{N_+}$ and some $Z\in\mathbb R^{N_-+N_+}$ with $z_i-z_j\ge 2L/3$ for all $i>j$. Then there exists $L_0>0$ only depending on $\vec c$, such that for $L>L_0\gg 1$ large enough, the function $v=(4-\partial_x^2)^{-1}u$ has got a unique point of local maximum (resp. minimum) ξ_i on $\Theta_{z_i}=[z_i-6.7,z_i+6.7]$ for any $1\le i\le N_+$ (resp. $-N_-\le i\le -1$). Moreover,

$$\Big\|u-\sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+}\varphi_{c_j}(\cdot-\xi_j)\Big\|_{\mathcal H}\le \Big\|u-\sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+}\varphi_{c_j}(\cdot-z_j)\Big\|_{\mathcal H}+O(e^{-L/4})\ . \tag{187}$$

and

$$\xi_i\in\mathcal V_i=[z_i-\ln\sqrt2,z_i+\ln\sqrt2],\quad \forall i\in[[-N_-,N_+]]\setminus\{0\}\ . \tag{188}$$

Finally, for any $(y_1,..,y_{N_+})\in\mathbb R^{N_+}$, such that

$$z_{-1}+L/4<y_1<z_1<y_2<z_2<\cdot\cdot<y_{N_+}<z_{N_+}$$

with $|y_i-z_j|\ge L/4$ for $(i,j)\in[[1,N_+]]^2$ it holds

$$\sup_{x\in]y_i,y_{i+1}[\setminus\Theta_{z_i}}\big(|u(x)|,|v(x)|,|v_x(x)|\big)\le\frac{c_1\wedge|c_{-1}|}{100}\ ,\quad i\in[[1,N_+]]\ , \tag{189}$$

where we set $y_{N_++1}=+\infty$.

Proof. Since $z_i - z_j \geq 2L/3$ for all $i > j$ it holds

$$\sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \rho_{c_j}(x - z_j) = \rho_{c_i}(x - z_i) + \|\vec{c}\|_1 O(e^{-L/4}), \quad \forall x \in [z_i - L/3, z_i + L/3] \,. \tag{190}$$

Therefore repeating the proof of Lemma 6 on each $[z_i - L/3, z_i + L/3]$, we obtain that, for L large enough, the function $v = (4 - \partial_x^2)^{-1} u$ has got a unique point of maximum (resp. minimum) ξ_i on $\Theta_{z_i} = [z_i - 6.7, z_i + 6.7]$ for any $1 \leq i \leq N_+$(resp. $-N_- \leq i \leq -1$) and that moreover $\xi_i \in \mathcal{V}_i$. In particular, $\xi_i - \xi_j \geq L/2$ for $i > j$ and thus applying (182) for the $z_i's$ and then the $\xi_i's$, (187) follows.

□

6.6. **Beginning of the proof of Theorem 2.** Let $\vec{c}$ and $A > 0$ be fixed and let $B = B(\vec{c}, A) \geq 1$ to be fixed at the end of this section. Let $\tilde{\alpha}_0$ be the minimum and $\tilde{L}_0$ be the maximum of respectively all the $\alpha_0(\vec{c})$ and all the $L_0(\vec{c})$ appearing in the preceding statements of Section 6. We set

$$\varepsilon_0 = \min\Big(\frac{10^{-20}}{B\tilde{K}}\Big(\frac{c_1 \wedge |c_{-1}|}{(1 + \|\vec{c}\|_1)(N_- + N_+)}\Big)^2,\ \tilde{\alpha}_0\Big) \,. \tag{191}$$

where $\tilde{K}$ is the constant depending on $\vec{c}$ that appear in Lemma 12. For $\alpha > 0$ we also set

$$T_\alpha = \max\Big(\frac{2^5}{c_1 \wedge |c_{-1}|)}(9 + \ln(\frac{\mathcal{A}_0}{\alpha^2})), 0\Big) \tag{192}$$

with

$$\mathcal{A}_0 = e^{27} \frac{(\|\vec{c}\|_1 + 1)^3}{(c_1 \wedge |c_{-1}|)^2}(1 + A) \,.$$

For $0 < \varepsilon < \varepsilon_0$ and $L > \tilde{L}_0$, we set $\alpha = B(\varepsilon + L^{-\frac{1}{8}})$. Since $\alpha \geq L^{-1/8}$, we have $\ln(1/\alpha^2) \leq \ln(L^{1/4})$ and thus

$$T_\alpha \leq \frac{2^5}{c_1 \wedge |c_{-1}|} \ln(L^{3/4}) \,,$$

as soon as $L \geq \mathcal{A}_0^4 \vee e^{36}$. Therefore we set

$$L_0 = \max(\varepsilon_0^{-8}, \mathcal{A}_0^4, \tilde{L}_0) \,. \tag{193}$$

According to Lemma 15, for $L \geq L_0$, this ensures that the solution $\mathbf{u}$ to (4) emanating from $\mathbf{u}_0 = \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j^0)$ satisfies

$$\big\|\mathbf{u}(t) - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(x - z_j^0 - c_j t)\big\|_{\mathcal{H}} \leq e^{-L/2} \leq L^{-1/8}, \quad \forall t \in [0, T_\alpha] \,.$$

On the other hand, according to the continuity with respect to initial data (see Proposition 2), for any $\varepsilon > 0$ there exists $\delta = \delta(A, \varepsilon, c) > 0$ such that for any $u_0 \in Y$ satisfying Hypothesis 1 and (17)-(18) with A and δ, it holds

$$\|u(t) - \mathbf{u}(t)\|_{\mathcal{H}} \leq \varepsilon, \quad \forall t \in [0, T_\alpha] \,,$$

where $u \in C(\mathbb{R}_+; H^1(\mathbb{R}))$ is the solution of the (D-P) equation emanating from u_0. Gathering the two above estimates we thus infer that

$$\big\|u(t) - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(x - z_j^0 - c_j t)\big\|_{\mathcal{H}} \leq \varepsilon + L^{-1/8}, \quad \forall t \in [0, T_\alpha] \,. \tag{194}$$

So let $u_0 \in Y \cap H^\infty(\mathbb{R})$ that satisfies Hypothesis 1 and (17)-(18) with A, δ and $L \geq L_0$. (194) together with the definitions (191)-(193) and Lemma 11 then ensure that

$$\big\|u(t) - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(x - z_j^0 - c_j t)\big\|_{L^\infty} < \frac{10^{-5}}{N_- + N_+}(c_1 \wedge |c_{-1}|), \quad \forall t \in [0, T_\alpha], \tag{195}$$

Applying Lemma 17 with the $z_j = z_j^0 + c_j t$ we obtain the existence of the local maxima (or minima) $\xi_j(t)$. Note that (188) ensures that $\xi_i(t) - \xi_j(t) \ge 2L/3$ for $i > j$ and (189) ensures that $\xi_i(t)$ is the only point of maximum (resp. point of minimum) of $v(t) = (4 - \partial_x^2)^{-1} u(t)$ on $[\xi_i(t) - L/4, \xi_i(t) + L/4]$ for $i \in [[1, N_+]]$ (resp. $i \in [[N_-, -1]]$).

By a continuity argument it remains to prove that for any $T \ge T_\alpha$, if

$$u(t) \in U\Big(2B(\varepsilon + L^{-1/8}), L/2\Big) \quad \text{on} \quad [0, T] \tag{196}$$

then there exists $\xi_{N_-}(T) < ..\xi_{-1}(T) < \xi_1(T) < ..\xi_{N_+}(T)$ with $\xi_i(T) - \xi_j(T) \ge 2L/3$ for $i > j$ such that

$$\big\| u(t) - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(x - \xi_j(T)) \big\|_{\mathcal{H}} \le B(\varepsilon + L^{-1/8}), \quad \forall t \in [0, T]\ . \tag{197}$$

and $\xi_i(T)$ is the only point of global maximum (resp. point of global minimum) of $v(t)$ on $[\xi_i(T) - L/4, \xi_i(T) + L/4]$ for $i \in [[1, N_+]]$ (resp. $i \in [[N_-, -1]]$). Now it is worth noticing that (196) together with the definitions (191)-(193) and Proposition 6 ensure that there exist $x_{N_-}(T) < ..x_{-1}(T) < x_1(T) < ..x_{N_+}(T)$ with $x_i(T) - x_j(T) \ge 3L/4$ for $i > j$ such that

$$\big\| u(t) - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(x - x_j(t)) \big\|_{\mathcal{H}} \le \tilde{K} B(\varepsilon + L^{-1/8}), \quad \forall t \in [0, T]\ . \tag{198}$$

and Lemma 11 together with (191)-(193) ensure that

$$\big\| u(t) - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(x - x_j(t)) \big\|_{L^\infty} \le \frac{10^{-5}}{N_- + N_+} (c_1 \wedge |c_{-1}|), \quad \forall t \in [0, T]\ .$$

Applying Lemma 17 with the $z_j = x_j(t)$ we obtain the existence of the local maximum (or minimum) $\xi_j(t)$. Note that (188) ensures that $\xi_i(t) - \xi_j(t) \ge 2L/3$ for $i > j$ and (189) ensures that $\xi_i(t)$ is the only point of global maximum (resp. point of global minimum) of $v(t) = (4 - \partial_x^2)^{-1} u(t)$ on $[\xi_i(t) - L/4, \xi_i(t) + L/4]$ for $i \in [[1, N_+]]$ (resp. $i \in [[N_-, -1]]$). Moreover, (187) and again Lemma 11 prove that for $L \ge L_0$ large enough

$$\big\| u(t) - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(x - \xi_j(t)) \big\|_{\mathcal{H}} \le 2\tilde{K} B(\varepsilon + L^{-1/8}) \tag{199}$$

and

$$\big\| u(t) - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(x - \xi_j(t)) \big\|_{L^\infty} \le \frac{10^{-5}}{N_- + N_+} (c_1 \wedge |c_{-1}|), \quad \forall t \in [0, T]\ .$$

Finally, Proposition 6 together with the definition (192) of T_α and (188) then ensure that

$$u(t, \cdot) - 6v(t, \cdot) \le \alpha^2 = (\varepsilon + L^{-1/8})^2 \quad \text{on} \quad [x_1(t) - 8, +\infty[\quad \forall t \in [T_\alpha, T]\ . \tag{200}$$

For the remaining of the proof we need the following localized versions of Lemmas 7-10, where the global functional E and F are replaced by their localized versions E_i and F_i.

6.7. **Localized estimates.** In the sequel we set

$$K = \sqrt{L}/8\ . \tag{201}$$

Let $x_{-N_-}(\cdot) < .. < x_{-1}(\cdot) < x_1(\cdot) < .. < x_{N_+}(\cdot)$ be the $N_- + N_+$ C^1-functions defined on $[0, T]$ (see (198)) and define the function $\Phi_i = \Phi_i(t, x)$, $i = 1, .., N_+$, by

$$\begin{cases} \Phi_{N_+}(t) = \Psi_{N_+, \sqrt{L}/8}(t) = \Psi_{\sqrt{L}/8}(\cdot - y_{N_+}(t)) \\ \Phi_i(t) = \Psi_{i, \sqrt{L}/8}(t) - \Psi_{i+1, \sqrt{L}/8}(t) = \Psi_{\sqrt{L}/8}(\cdot - y_i(t)) - \Psi_{\sqrt{L}/8}(\cdot - y_{i+1}(t)), \quad i = 1, ..., N_+ - 1, \end{cases} \tag{202}$$

where $\Psi_{i,K}$ and the y_i's are defined in Section 6.2 (130)-(135). It is easy to check that the Φ_i's are positive functions and that $\displaystyle\sum_{i=1}^{N_+}\Phi_i \equiv \Psi_{1,\sqrt{L}/8}$. Since $L \geq L_0 \geq 1$, (201) and (131) ensure that Φ_i satisfies for $i \in \{1, ..., N_+\}$,

$$\big|1-\Phi_i\big| \leq 2e^{-\sqrt{L}} \text{ on } \Big]y_i + \frac{L}{8}, y_{i+1} - \frac{L}{8}\Big[, \tag{203}$$

and

$$\big|\Phi_i\big| \leq 2e^{-\sqrt{L}} \text{ on } \mathbb{R}\backslash \Big]y_i - \frac{L}{8}, y_{i+1} + \frac{L}{8}\Big[\ , \tag{204}$$

where we set $y_{N_+ + 1} := +\infty$.
It is worth noticing that, somehow, $\Phi_i(t)$ takes care of only the ith bump of $u(t)$. We will use the following localized version of E and F defined for $i \in \{1, .., N_+\}$, by

$$E_i(t) = \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2)\Phi_i(t) \text{ and } F_i(t) = \int_{\mathbb{R}} \left(-v_{xx}^3 + 12vv_{xx}^2 - 48v^2 v_{xx} + 64v^3\right)\Phi_i(t)\ . \tag{205}$$

In the statement of the four following lemmas we fix the time. This corresponds to fix $x_{-N_-} < .. < x_{-1} < x_1 < .. < x_{N_+}$ with $x_i - x_j > 3L/4$ for $i > j$ such that

$$\Big\|u(t) - \sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \varphi_{c_j}(x - x_j)\Big\|_{\mathcal{H}} \leq \tilde{K}B(\varepsilon + L^{-1/8}) \tag{206}$$

and to fix $(y_1, .., y_{N_+}) \in \mathbb{R}^{N_+}$, such that

$$x_{-1} + L/4 < y_1 < x_1 < y_2 < x_2 < \cdot\cdot < y_{N_+} < x_{N_+} < y_{N_+ + 1} = +\infty$$

with $|y_i - x_j| \geq L/4$ for $(i, j) \in [[1, N_+]]^2$. In particular, E_i and F_i do not depend on time.
For $i = 1, ..., N_+$, we set $\Omega_i =]y_i - L/8, y_{i+1} + L/8[$, the interval in which the mass of each peakon φ_{c_i} (and smooth peakon ρ_{c_i}) is concentrated. One can see that

$$\sum_{\substack{j=-N_-\\ j\neq 0}}^{N_+} \rho_{c_j}(x - x_j) = \rho_{c_i}(x - x_i) + O(e^{-L/4}), \quad \forall x \in \Omega_i, \tag{207}$$

and that $\rho_{c_i}(x - x_i) = O(e^{-L/4})$ for all $x \in \mathbb{R}\backslash\Omega_i$. We will decompose Ω_i as in Section 5 by setting

$$\Theta_i = [x_i - 6.7, x_i + 6.7], \text{ where } 6.7 \simeq \ln\left(\frac{20}{20 - \sqrt{399}}\right) > \ln\sqrt{2}, \text{ with } \rho_{c_i}(\pm 6.7) \simeq c_i/2400. \tag{208}$$

Lemma 18 (See [10])**.** *Let $u \in L^2(\mathbb{R})$ satisfying* (206)*. Denote by $M_i = \max_{x\in\Theta_i} v(x) = v(\xi_i)$ and define for $i = 1, ..., N_+$ the function g_i by*

$$g_i(x) = \begin{cases} 2v(x) + v_{xx}(x) - 3v_x(x), \ \forall x < \xi_i, \\ 2v(x) + v_{xx}(x) + 3v_x(x), \ \forall x > \xi_i. \end{cases} \tag{209}$$

Then it holds

$$\int_{\mathbb{R}} g_i^2(x)\Phi_i(x)dx = E_i(u) - 12M_i^2 + \|u\|_{\mathcal{H}}^2 O(L^{-1/2}), \tag{210}$$

and

$$\begin{aligned} \int_{\mathbb{R}} g_i^2(x)\Phi_i(x)dx &= E_i(u - \varphi_{c_i}(\cdot - \xi_i)) - 12\left(\frac{c_i}{6} - M_i\right)^2 + \|u - \varphi_{c_i}(\cdot - \xi_i)\|_{\mathcal{H}}^2 O(L^{-1/2}) \\ &\leq O(\|u - \varphi_{c_i}(\cdot - \xi_i)\|_{\mathcal{H}}^2) \end{aligned} \tag{211}$$

Proof. The proof is similar to the one of Lemma 4.3 in [10] using (203) and $|\Phi'| + |\Phi''| \leq O(L^{-1/2})$ since $K = \sqrt{L}/8$. The second identity follows as (106) in Lemma 7. ☐

Lemma 19 (See [10])**.** *Let $u \in L^2(\mathbb{R})$ satisfying* (206)*. Denote by $M_i = \max_{x\in\Theta_i} v(x) = v(\xi_i)$ and define for $i = 1, ..., N_+$ the function h_i by*

$$h_i(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \xi_i, \\ -v_{xx} + 6v_x + 16v, & x > \xi_i. \end{cases} \tag{212}$$

Then, it holds

$$\int_{\mathbb{R}} h_i(x) g_i^2(x)\Phi_i(x)dx = F_i(u) - 144M_i^3\Phi_i(\xi_i) + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}). \tag{213}$$

Proof. The proof is similar to the one of Lemma 4.3 in [10] using the fact that $K = \sqrt{L}/8$ and thus $|\Phi'| + |\Phi''| \le O(L^{-1/2})$. $\square$

Lemma 20 (Connection between the conservation laws E_i and F_i)**.** *Let $u \in L^2(\mathbb{R})$ satisfying Hypothesis 1 and* (206)*.*
If

$$u - 6v \le \alpha^2 \quad on \quad [x_1 - 8, +\infty[\tag{214}$$

and for $i \in [[1, N_+]]$,

$$\sup_{x\in]y_i - \frac{L}{8}, y_{i+1} + \frac{L}{8}[\setminus \Theta_{\xi_i}} (|u(x)|, |v(x)|, |v_x(x)|) \le \frac{c_i}{100}\ , \tag{215}$$

then it holds

$$F_i(u) \le 18M_iE_i(u) - 72M_i^3 + O(\alpha^4) + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}), \quad i = 1, ..., N_+. \tag{216}$$

Proof. Recall that, according to Subsection 6.6, $v = (4 - \partial_x^2)^{-1}u$ has a got a unique global maximum ξ_i on $\xi_i \in]y_i + L/8, y_{i+1} - L/8[$ for $i \in [[1, N_+]]$. , it follows from (203) that $\Phi_i(\xi_i) = 1 + O(e^{-\sqrt{L}})$. Combining this with $K = \sqrt{L}/8$, (210) and (213) one may deduce that

$$\int_{\mathbb{R}} g_i^2(x)\Phi_i(x)dx = E_i(u) - 12M_i^2 + \|u\|_{\mathcal{H}}^2 O(L^{-1/2}), \tag{217}$$

and

$$\int_{\mathbb{R}} h_i(x)g_i^2(x)\Phi_i(x)dx = F_i(u) - 144M_i^3 + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}). \tag{218}$$

Now, in view of (204) and (39) it holds

$$\left| \int_{\mathbb{R}\setminus\Omega_i} h_i(x)g_i^2(x)\Phi_i(x)dx \right| = \|u\|_{\mathcal{H}}^2 (\|u\|_{L^\infty} + \|u\|_{\mathcal{H}}) O(e^{-\sqrt{L}/8}) = \|u\|_{\mathcal{H}}^3 O(e^{-\sqrt{L}/8})\ .$$

It thus remains to show that the function h_i defined in Lemma 19 satisfies $h_i \le 18M_i + \alpha^2$ on Ω_i. We divide Ω_i into three intervals. If $x \in \Omega_i \setminus \Theta_{\xi_i}$, then using (214), it holds

$$h_i(x) \le |u(x)| + 6|v_x(x)| + 12|v(x)| \le \frac{19c_i}{100} \le 18M_i. \tag{219}$$

If $\xi_i - 6.7 < x < \xi_i$, then $v_x \ge 0$ and using that $u - 6v \le \alpha^2$, we get

$$h_i(x) \le 18M_i + \alpha^2. \tag{220}$$

If $\xi_i < x < \xi_i + 6.7$, then $v_x \le 0$ and using that $u - 6v \le \alpha^2$, we get

$$h_i(x) \le 18M_i + \alpha^2.$$

Combining (217), (204), and (218), one deduce that

$$\begin{aligned} F_i(u) - 144M_i^3 &= \int_{\mathbb{R}} h_i(x)g_i^2(x)\Phi_i(x)dx + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\ &= \int_{\Omega_i} h_i(x)g_i^2(x)\Phi_i(x)dx + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\ &\le 18M_i\Big(E_i(u) - 12M_i^2\Big) + O(\alpha^4) + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}), \end{aligned}$$

that completes the proof of the lemma. $\square$

Lemma 21. *Let $u_0 \in Y$ satisfying Hypothesis 1 and* (17)-(19)*. It holds*

$$\big|E_i(u_0) - E(\varphi_{c_i})\big| + \big|F_i(u_0) - F(\varphi_{c_i})\big| \le O(\varepsilon^4) + O(e^{-\sqrt{L}}), \qquad i \in [[-N_-, N_+]] \setminus \{0\}. \tag{221}$$

Proof. It follows easily from (17)-(19), (205), the exponential decay of φ_{c_i} and Φ_i and the choice $K = \sqrt{L}/8$ (see Lemma 4.7 in [10] for details). $\square$

6.8. **End of the proof of Theorem 2.** For $i \in [[1, N_+]$ we set $M_i = v\big(T, \xi_i(T)\big)$ and $\delta_i = c_i/6 - M_i$. It is worth recalling that Lemma 17 ensures that for $1 \le i \le N_+$, $v(T, \xi_i(T)) = \max_{[y_i(T), y_{i+1}(T)]} v(T, \cdot)$, where the y_i's are defined in (136). For a function $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$, we set

$$\Delta_0^T f = f(T) - f(0).$$

Summing (216) over $i \in \{1, ..., N_+\}$, we get

$$\sum_{i=1}^{N_+} \Delta_0^T F_i(u) \le 18 \sum_{i=1}^{N_+} M_i \Delta_0^T E_i(u) + \sum_{i=1}^{N_+} \Big[-72 M_i^3 + 18 M_i E_i(u_0) - F_i(u_0) \Big] + O(\alpha^4) + O(L^{-1/2})$$

that can be rewritten after some computations as

$$\begin{aligned}\sum_{i=1}^{N_+} \Big[M_i^3 - \frac{1}{4} M_i E(\varphi_{c_i}) + \frac{1}{72} F(\varphi_{c_i}) \Big] &\le \frac{1}{4} \sum_{i=1}^{N_+} \Big(M_i \Delta_0^T E_i(u) - \frac{1}{18} \Delta_0^T F_i(u) \Big) \\ &\quad + \frac{1}{4} \sum_{i=1}^{N_+} M_i \big|E_i(u_0) - E(\varphi_{c_i})\big| + \frac{1}{72} \sum_{i=1}^{N_+} \big|F_i(u_0) - F(\varphi_{c_i})\big| + O(\alpha^4) + O(L^{-1/2}).\end{aligned} \tag{222}$$

Using Abel transformation, the fact that $E(\varphi_{c_i}) = c_i^2/3$, $F(\varphi_{c_i}) = 2c_i^3/3$ and definition (134), (noticing that $0 \le 1/18M_1 < 2/3c_1$) we obtain

$$\begin{aligned}\sum_{i=1}^{N_+} \delta_i^2 \left[\frac{c_i}{2} - \delta_i \right] &\le \frac{1}{4} M_1 \Delta_0^T \mathcal{J}_{1, \frac{1}{18M_1}, K} + \frac{1}{4} \sum_{i=2}^{N_+} \Big(M_i - M_{i-1} \Big) \Delta_0^T \mathcal{J}_{i,0,K} \\ &\quad + \frac{1}{4} \sum_{i=1}^{N_+} M_i \big|E_i(u_0) - E(\varphi_{c_i})\big| + \frac{1}{72} \sum_{i=1}^{N_+} \big|F_i(u_0) - F(\varphi_{c_i})\big| + O(\alpha^4) + O(L^{-1/2}).\end{aligned} \tag{223}$$

Now, in view of Lemma (199) and (207)

$$M_i = \frac{c_i}{6} + O(e^{-L/4}) + O(\alpha),$$

and thus for $0 < \alpha < \alpha_0(\vec{c}) \ll 1$ small enough and $L > L_0 \gg 1$ large enough, it holds

$$0 < M_1 < ... < M_{N_+} \qquad \text{and} \qquad \delta_i < c_i/4, \qquad \text{with } i = 1, ..., N_+. \tag{224}$$

Combining (221), (223), (224) and (137), we obtain

$$\sum_{i=1}^{N_+} |c_i \delta_i| \le O(\varepsilon^2 + L^{-1/4}) \, . \tag{225}$$

Now, it is again crucial to note that (D-P) is invariant by the change of unknown $u(t,x) \mapsto \tilde{u}(t,x) = -u(t,-x)$. As in the proof of Proposition 5 it is clear that $\tilde{u}(0,\cdot) = -u_0(-\cdot)$ satisfies Hypothesis 1 with $\tilde{x}_0 = -x_0$ and then $\tilde{x}_0(t) = -x_0(t)$ for all $t \ge 0$. $\tilde{u}$ satisfies (198) on $[0,T]$ with N_- and N_+ respectively replaced by $\tilde{N}_- = N_+$ and $\tilde{N}_+ = N_-$, $x_i(t)$ replaced by $\tilde{x}_i(t) = -x_{-i}$ and c_i replaced by $\tilde{c}_i = -c_{-i}$. Also we notice that the definition of T_α is symmetric in c_1 and $-c_{-1}$ so that $\tilde{u}$ also satisfies (200) with v replaced by $\tilde{v}$ and $x_1(t)$ replaced by $\tilde{x}_1(t)$. Therefore, applying the above procedure for $\tilde{u}$ we obtain as well that

$$\sum_{i=1}^{N_-} |c_{-i}(c_{-i}/6 - M_{-i})| = \sum_{i=1}^{\tilde{N}_+} |\tilde{c}_i(\tilde{c}_i/6 - \tilde{M}_i)| \le O(\varepsilon^2 + L^{-1/4}) \, , \tag{226}$$

with $\tilde{M}_i = -M_{-i}$ where $M_{-i} = v(T, \xi_{-i}) = \min_{\Omega_i} v(T, \cdot)$.

To conclude the proof we need the following estimate on the left-hand side member of (182).

Lemma 22. *For any $u_0 \in L^2(\mathbb{R})$ satisfying* (18)-(19)*, it holds*

$$\Big|E(u_0) - \sum_{i=-N_-}^{N_+} E(\varphi_{c_i})\Big| \leq O(\varepsilon^4) + O(e^{-L/2}) \tag{227}$$

Proof. It follows easily from (18)-(19) and the exponential decay of $\rho_{c_i} = (4-\partial_x^2)^{-1}\varphi_{c_i}$ (see Lemma 4.7 in [10] for details). □

Gathering (225)-(226), Lemma 16 and (227) with $\delta \leq \varepsilon^4$ we obtain that there exists $C > 0$ only depending on $\vec{c}$ such that

$$\Bigg\|u(T) - \sum_{\substack{i=-N_-\\ i\neq 0}}^{N_+} \varphi_{c_i}(\cdot - \xi_i(T))\Bigg\|_{\mathcal{H}} \leq C(\varepsilon + L^{-1/8}),$$

and (196) holds by choosing $B = C \vee 1$.

7. Appendix

7.1. **Proof of Lemma 13.** Identity (138) is a simplified version of the one derived in [10] Appendix 4.4. We start by applying the operator $(4-\partial_x^2)^{-1}(\cdot)$ on the both sides of equation (4) and using the fact that

$$(4-\partial_x^2)^{-1}(1-\partial_x^2)^{-1}(\cdot) = \frac{1}{3}(1-\partial_x^2)^{-1}(\cdot) - \frac{1}{3}(4-\partial_x^2)^{-1}(\cdot), \tag{228}$$

we infer that $v = (4-\partial_x^2)^{-1}u$ satisfies

$$v_t + \frac{1}{2}h_x = 0, \tag{229}$$

where $h = (1-\partial_x^2)^{-1}u^2$. With this identity in hand one may check that

$$4\frac{d}{dt}\int_{\mathbb{R}} v^2 g dx = 8\int_{\mathbb{R}} v v_t g dx = -4\int_{\mathbb{R}} v h_x g dx\ .$$

Since $\partial_x^2(1-\partial_x^2)^{-1}(\cdot) = -(\cdot) + (1-\partial_x^2)^{-1}(\cdot)$, (229) then leads to

$$\begin{aligned}5\frac{d}{dt}\int_{\mathbb{R}} v_x^2 g dx &= 10\int_{\mathbb{R}} v_x v_{xt} g dx = -5\int_{\mathbb{R}} v(1-\partial_x^2)^{-1}\partial_x^2(u^2) g dx = 5\int_{\mathbb{R}} u^2 v_x g dx - 5\int_{\mathbb{R}} v_x h g dx\\ &= 5\int_{\mathbb{R}} u^2 v_x g dx + 5\int_{\mathbb{R}} v h_x g dx + 5\int_{\mathbb{R}} v h g' dx\ .\end{aligned}$$

Moreover in the same way one may write

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} v_{xx}^2 g dx &= 2\int_{\mathbb{R}} v_{xx} v_{xxt} g dx = -\int_{\mathbb{R}} v_{xx}(1-\partial_x^2)^{-1}\partial_x^3(u^2) g dx = \int_{\mathbb{R}} \partial_x(u^2) v_{xx} g dx - \int_{\mathbb{R}} v_{xx} h_x g dx\\ &= A_1 + A_2,\end{aligned}$$

where since $v_{xx} = u - 4v$, it holds

$$A_1 = -\int_{\mathbb{R}} \partial_x(u^2) u g dx + 4\int_{\mathbb{R}} \partial_x(u^2) v g dx = \frac{2}{3}\int_{\mathbb{R}} u^3 g' dx - 4\int_{\mathbb{R}} u^2 v_x g dx - 4\int_{\mathbb{R}} u^2 v g' dx$$

and

$$A_2 = \int_{\mathbb{R}} v_x(1-\partial_x^2)^{-1}\partial_x^2(u^2) g dx + \int_{\mathbb{R}} v_x h_x g' dx = -\int_{\mathbb{R}} u^2 v_x g dx + \int_{\mathbb{R}} v_x h g dx + \int_{\mathbb{R}} v_x h_x g' dx.$$

Gathering the above identities, (138) follows. We now concentrate on the proof of (139). Using equation (8) one may write

$$\frac{d}{dt}\int_{\mathbb{R}} u^3 g dx = -\frac{3}{2}\int_{\mathbb{R}} u^2 (u^2)_x g dx - \frac{9}{2}\int_{\mathbb{R}} u^2 h_x g dx = I_1 + I_2. \tag{230}$$

First, by integration by parts one may have

$$I_1 = \frac{3}{4}\int_{\mathbb{R}} u^4 g' dx.$$

Second, substituting u^2 by $h - h_{xx}$ and integrating by parts we get

$$I_2 = -\frac{9}{2}\int_{\mathbb{R}} hh_x g dx + \frac{9}{2}\int_{\mathbb{R}} h_x h_{xx} g dx = \frac{9}{4}\int_{\mathbb{R}} (h^2 - h_x^2) g' dx$$

that proves (139). Finally, (140) can be deduced directly from (7) by integrating by parts in the following way :

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} yg\, dx &= -\int_{\mathbb{R}} \partial_x(yu)g - 3\int_{\mathbb{R}} yu_x g \\ &= \int_{\mathbb{R}} yug' - 3\int_{\mathbb{R}} (u - u_{xx})u_x g \\ &= \int_{\mathbb{R}} yug' + \frac{3}{2}\int_{\mathbb{R}} (u^2 - u_x^2)g' \ . \end{aligned}$$

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